Вежбе

(14.) $(a>0)$ $\int \frac{dx}{x^2-a^2} = ?$

$x^2-a^2=(x-a)(x+a)$

$\frac{1}{x^2-a^2} = \frac{1}{(x-a)(x+a)} = \frac{A}{x-a} + \frac{B}{x+a} = \frac{A(x+a)+B(x-a)}{(x+a)(x-a)} \Longrightarrow A(x+a)+B(x-a) \equiv 1$

($A, B \to$ константе)

$(A+B)x + (A-B)a \equiv 1$

$(A+B)x + (A-B)a = 0 \cdot x + 1$

$A+B=0$

$(A-B)a = 1$

$B=-A$

$1 = (A-A)x + (A+A)a$

$2A \cdot a = 1$

$\boxed{A = \frac{1}{2a},\ B = -\frac{1}{2a}}$

$\int \frac{dx}{x^2-a^2} = \int \left(\frac{A}{x-a} + \frac{B}{x+a}\right) dx =$

$= \int \frac{1}{2a} \cdot \frac{1}{x-a} dx + \int -\frac{1}{2a} \cdot \frac{1}{x+a} dx =$

$= \frac{1}{2a} \int \frac{dx}{x-a} - \frac{1}{2a} \int \frac{dx}{x+a} = \frac{1}{2a}\left(\ln|x-a| - \ln|x+a|\right) + C =$

$= \frac{1}{2a} \ln\left|\frac{x-a}{x+a}\right| + C$

(15.) I $\int (2x+1)^2 dx = \int t^2 \frac{dt}{2} = \frac{1}{2} \cdot \frac{t^3}{3} = \frac{1}{6}(2x+1)^3 + C = \frac{1}{6}(8x^3 + 12x^2 + 6x + 1) + C =$

$2x+1=t$

$2dx=dt$

$dx=\frac{dt}{2}$

$= \frac{4}{3}x^3 + 2x^2 + x + \frac{1}{6} + C$ ← константе (исправна су оба)

II $\int (2x+1)^2 dx = \int 4x^2 dx + \int 4x dx + \int dx = 4 \cdot \frac{x^3}{3} + 4 \cdot \frac{x^2}{2} + x + C = \frac{4}{3}x^3 + 2x^2 + x + C$

-Парцијална (делимична) интеграција-

$\boxed{\int u\,dv = uv - \int v\,du} = \int u(x) \cdot v'(x) dx = u(x) \cdot v(x) - \int v(x) \cdot u'(x) dx$

1. $\int xe^x dx = xe^x - \int e^x dx = xe^x - e^x + C = e^x(x-1)+C$

$u = x \quad e^x dx = dv$

$du = dx \quad \int e^x dx = \int dv$

$e^x = v$

Погрешан начин!: $\int xe^x dx = \frac{x^2}{2}\cdot e^x - \int \frac{x^2}{2} e^x dx = \ldots = ?$

$e^x = u \quad x dx = dv$

$e^x dx = du \quad \frac{x^2}{2} = v$

2. $\int \ln x dx = x\ln x - \int x \frac{dx}{x} = x\ln x - x + C = x(\ln x - 1) + C$

$\ln x = u \quad dx = dv$

$\frac{dx}{x} = du \quad \int dx = \int dv$

$x = v$

3. $\int \ln^2 x dx = x\cdot \ln^2 x - \int x \cdot \frac{2\ln x}{x} dx = x\cdot\ln^2 x - 2\int \ln x dx = x\cdot \ln^2 x - 2(x\cdot \ln x - x) =$

$\ln^2 x = u \quad dx = dv$

$\frac{2\ln x}{x} dx = du \quad \int dx = \int dv$

$x = v$

$= x\cdot(\ln^2 x - 2\ln x + 2) + C$

4. $\int \operatorname{arctg} x dx = x\operatorname{arctg} x - \int x\frac{dx}{x^2+1} = x\operatorname{arctg} x - \int \frac{dt}{2t} = x\operatorname{arctg} x - \frac{1}{2}\ln|t| + C =$

$\operatorname{arctg} x = u \quad dx = dv$

$\frac{dx}{x^2+1} = du \quad \int dx = \int dv$

$x = v$

$x^2+1 = t$

$2x dx = dt$

$x dx = \frac{dt}{2}$

$= x\operatorname{arctg} x - \frac{1}{2}\ln(x^2+1) + C = x\cdot \operatorname{arctg} x - \ln\sqrt{x^2+1} + C$

5. $\int \arcsin x dx = x\cdot \arcsin x - \int x\cdot \frac{dx}{\sqrt{1-x^2}} = x\arcsin x + \frac{1}{2}\int \frac{dt}{\sqrt{t}} = x\arcsin x +$

$\arcsin x = u \quad dx = dv$

$\frac{dx}{\sqrt{1-x^2}} = du \quad \int dx = \int dv$

$x = v$

$1-x^2 = t$

$-2x dx = dt$

$x dx = -\frac{dt}{2}$

$+\frac{1}{2}\int t^{-\frac{1}{2}} dt = x\cdot \arcsin x + \cancel{\frac{1}{2}}\cdot \frac{t^{\frac{1}{2}}}{\cancel{\frac{1}{2}}} + C = x\cdot \arcsin x + \sqrt{1-x^2} + C$

6. $\int x^2 e^x dx = x^2 \cdot e^x - \int e^x \cdot 2x dx = x^2 \cdot e^x - 2\int x \cdot e^x dx =$

$u = x^2 \quad e^x dx = dv$
$du = 2x dx \quad \int e^x dx = \int dv$
$e^x = v$

$x = u_1 \quad e^x dx = dv_1$
$dx = du_1 \quad \int e^x dx = \int dv_1$
$e^x = v_1$

$= x^2 \cdot e^x - 2\left(e^x \cdot x - \int e^x dx\right) = x^2 \cdot e^x - 2\left(e^x \cdot x - e^x\right) + C = e^x\left(x^2 - 2x + 2\right) + C$

7. $\int e^{2x} \sin 3x dx = -\frac{1}{3}\cos 3x \cdot e^{2x} - \int -\frac{1}{3}\cos 3x \cdot 2e^{2x} dx = \frac{1}{3}\cos 3x \cdot e^{2x} + \frac{2}{3}\int \cos 3x \cdot e^{2x} dx =$

$e^{2x} = u \quad \sin 3x dx = dv$
$2e^{2x} dx = du \quad \int \sin 3x dx = \int dv$
$-\frac{1}{3}\cos 3x = v$

$e^{2x} = u \quad \cos 3x dx = dv$
$2e^{2x} dx = du \quad \int \cos 3x dx = \int dv$
$\frac{1}{3}\sin 3x = v$

$= -\frac{1}{3}e^{2x} \cdot \cos 3x + \frac{2}{3}\left(\frac{1}{3}e^{2x} \cdot \sin 3x - \int \frac{1}{3}\sin 3x \cdot 2e^{2x} dx\right) =$

$= -\frac{1}{3}e^{2x} \cdot \cos 3x + \frac{2}{3}\left(\frac{1}{3}e^{2x} \sin 3x - \frac{2}{3}\int e^{2x} \sin 3x dx\right)$

$\int e^{2x} \sin 3x dx = I$

$I = -\frac{1}{3}e^{2x} \cdot \cos 3x + \frac{2}{9}e^{2x} \sin 3x - \frac{4}{9} \cdot I$

$\frac{13}{\cancel{9}_3} I = \frac{1}{\cancel{3}} e^{2x}\left(-\cos 3x + \frac{2}{3}\sin 3x\right)$

$I = \frac{3e^{2x}}{13}\left(-\cos 3x + \frac{2}{3}\sin 3x\right)$

$I = \frac{1}{13}e^{2x}\left(2\sin 3x - 3\cos 3x\right)$

8. За бенду: $(a, b \in \mathbb{R})$

а) $\int e^{ax} \sin bx dx = ?$

б) $\int e^{ax} \cos bx dx = ?$

а) $\int e^{ax} \sin bx dx = -\frac{1}{b}\cos bx \cdot e^{ax} - \int -\frac{1}{b}\cos bx \cdot ae^{ax} dx = -\frac{1}{b}\cos bx \, e^{ax} + \frac{a}{b}\int e^{ax} \cos bx dx =$

$e^{ax} = u \quad \sin bx dx = dv$
$ae^{ax} dx = du \quad \int \sin bx dx = \int dv$
$-\frac{1}{b}\cos bx = v$

$e^{ax} = u \quad \cos bx dx = dv$
$ae^{ax} dx = du \quad \int \cos bx dx = \int dv$
$\frac{1}{b}\sin bx = v$

$$=-\frac{1}{b}\cos bx\cdot e^{ax}+\frac{a}{b}\left(\frac{1}{b}\sin bx\cdot e^{ax}-\int\frac{1}{b}\sin bx\cdot a\cdot e^{ax}dx\right)=$$

$$=-\frac{1}{b}\cos bx\, e^{ax}+\frac{a}{b}\left(\frac{1}{b}\sin bx\cdot e^{ax}-\frac{a}{b}\int e^{ax}\sin bx\,dx\right)=$$

$$\int e^{ax}\sin bx\,dx=I$$

$$I=-\frac{1}{b}\cos bx\cdot e^{ax}+\frac{a}{b^2}\cdot\sin bx\cdot e^{ax}-\frac{a^2}{b^2}\cdot I$$

$$I\left(\frac{a^2+b^2}{\cancel{b^2}}\right)=\frac{1}{\cancel{b}}e^{ax}\left(\frac{a}{b}\cdot\sin bx-\cos bx\right)$$

$$I=\frac{b}{a^2+b^2}\cdot e^{ax}\left(\frac{a}{b}\sin bx-\cos bx\right)$$

$$I=\frac{e^{ax}}{a^2+b^2}(a\cdot\sin bx-b\cdot\cos bx)$$

б) $$\int e^{ax}\cos bx\,dx=\frac{1}{b}\sin bx\cdot e^{ax}-\int\frac{1}{b}\sin bx\; ae^{ax}dx=\frac{1}{b}e^{ax}\cdot\sin bx-\frac{a}{b}\int e^{ax}\sin bx\,dx=$$

$e^{ax}=u$ $\quad$ $\cos bx\,dx=dv$

$ae^{ax}dx=du$ $\quad$ $\int\cos bx\,dx=\int dv$

$\frac{1}{b}\sin bx=v$

$e^{ax}=u$ $\quad$ $\sin bx\,dx=dv$

$ae^{ax}dx=du$ $\quad$ $\int\sin bx\,dx=\int dv$

$-\frac{1}{b}\cos bx=v$

$$=\frac{1}{b}e^{ax}\cdot\sin bx-\frac{a}{b}\left(-\frac{1}{b}\cos bx\cdot e^{ax}-\int-\frac{1}{b}\cos bx\cdot ae^{ax}dx\right)=$$

$$=\frac{1}{b}e^{ax}\sin bx+\frac{a}{b}\left(\frac{1}{b}e^{ax}\cos bx-\int\frac{a}{b}e^{ax}\cos bx\,dx\right)=$$

$$=\frac{1}{b}e^{ax}\sin bx+\frac{a}{b^2}\left(e^{ax}\cos bx-a\int e^{ax}\cos bx\,dx\right)$$

$$\int e^{ax}\cos bx\,dx=I$$

$$I=\frac{1}{b}e^{ax}\sin bx+\frac{a}{b^2}e^{ax}\cos bx-\frac{a^2}{b^2}\cdot I$$

$$I\cdot\frac{b^2+a^2}{\cancel{b^2}}=\frac{e^{ax}}{\cancel{b}}\left(\sin bx+\frac{a}{b}\cos bx\right)$$

$$I=\frac{b}{a^2+b^2}\cdot e^{ax}\left(\sin bx+\frac{a}{b}\cos bx\right)$$

$$I=\frac{e^{ax}}{a^2+b^2}(a\cdot\cos bx+b\cdot\sin bx)$$