Брзина и убрзање тачке у криволинијским координатама

$$\vec{r} = \vec{r}(q_1, q_2, q_3), \tag{1}$$

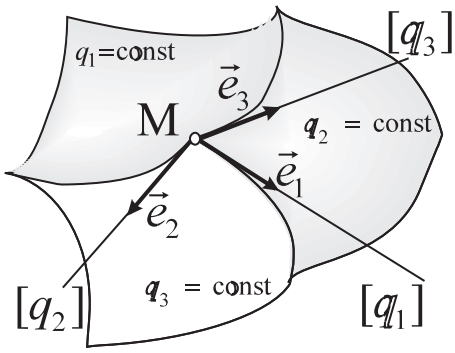
при чему се кретање тачке задаје једначинама

$$q_1 = q_1(t), \qquad q_2 = q_2(t) \quad \text{и} \quad q_3 = q_3(t). \tag{2}$$

$$x = x(q_1, q_2, q_3), \quad y = y(q_1, q_2, q_3) \quad \text{и} \quad z = z(q_1, q_2, q_3). \tag{3}$$

$$x = x(q_1, q_2^0, q_3^0), \qquad y = y(q_1, q_2^0, q_3^0) \quad \text{и} \quad z = z(q_1, q_2^0, q_3^0). \tag{4}$$

На исти начин могу се одредити и једначине координатних линија координата q_2 и q_3 у облику $\vec{r} = \vec{r}(q_1^0, q_2, q_3^0)$ и $\vec{r} = \vec{r}(q_1^0, q_2^0, q_3)$.



Слика 1

Координатна површ (q_1, q_2) одређена је следећим једначинама:

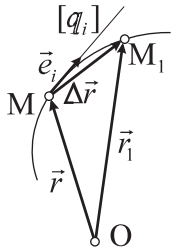
$$x = x(q_1, q_2, q_3^0), \qquad y = y(q_1, q_2, q_3^0) \quad \text{и} \quad z = z(q_1, q_2, q_3^0), \tag{5}$$

$$\frac{\partial \vec{r}}{\partial q_i} = \frac{\partial x}{\partial q_i}\vec{\imath} + \frac{\partial y}{\partial q_i}\vec{\jmath} + \frac{\partial z}{\partial q_i}\vec{k}, \qquad (i = 1, 2, 3) \tag{6}$$

$$\left|\frac{\partial \vec{r}}{\partial q_i}\right| = \sqrt{\frac{\partial x}{\partial q_i}^2 + \frac{\partial y}{\partial q_i}^2 + \frac{\partial z}{\partial q_i}^2} = h_i, \qquad (i = 1, 2, 3) \tag{7}$$

$$\vec{e}_i = \frac{1}{h_i}\frac{\partial \vec{r}}{\partial q_i}. \tag{8}$$

$$\vec{e}_i \cdot \vec{e}_j \;=\; \delta_{ij} \;=\; \begin{cases} 0 & \text{ако је } i \neq j, \\ 1 & \text{ако је } i = j, \end{cases} \qquad (i,j = 1,2,3) \tag{9}$$



$$\vec{r} \;=\; \vec{r}[q_1(t), q_2(t), q_3(t)], \tag{10}$$

$$\overrightarrow{V} \;=\; \frac{d\vec{r}}{dt} \;=\; \frac{\partial \vec{r}}{\partial q_1}\dot{q}_1 + \frac{\partial \vec{r}}{\partial q_2}\dot{q}_2 + \frac{\partial \vec{r}}{\partial q_3}\dot{q}_3 \;=\; \sum_{i=1}^{3} \frac{\partial \vec{r}}{\partial q_i}\dot{q}_i. \tag{11}$$

$$\frac{\partial \vec{r}}{\partial q_i} \;=\; h_i\, \vec{e}_i. \qquad (i = 1,2,3) \tag{12}$$

Слика 2

$$\overrightarrow{V} \;=\; \dot{q}_1 h_1 \vec{e}_1 + \dot{q}_2 h_2 \vec{e}_2 + \dot{q}_3 h_3 \vec{e}_3 \;=\; \sum_{i=1}^{3} \dot{q}_i h_i \vec{e}_i. \tag{13}$$

$$V \;=\; \sqrt{\dot{q}_1^2 h_1^2 + \dot{q}_2^2 h_2^2 + \dot{q}_3^2 h_3^2} \;=\; \sqrt{\sum_{i=1}^{3} \dot{q}_i^2 h_i^2} \;=\; \sqrt{\sum_{i=1}^{3} V_{q_i}^2}, \tag{14}$$

$$V_{q_i} \;=\; \dot{q}_i h_i \qquad (i = 1,2,3) \tag{15}$$

$$a_{q_i} \;=\; \vec{a} \cdot \vec{e}_i \;=\; \frac{1}{h_i}\frac{d\overrightarrow{V}}{dt} \cdot \frac{\partial \vec{r}}{\partial q_i}, \qquad (i = 1,2,3) \tag{16}$$

$$h_i a_{q_i} \;=\; \frac{d\overrightarrow{V}}{dt} \cdot \frac{\partial \vec{r}}{\partial q_i} \;=\; \frac{d}{dt}(\overrightarrow{V} \cdot \frac{\partial \vec{r}}{\partial q_i}) - \overrightarrow{V} \cdot \frac{d}{dt}(\frac{\partial \vec{r}}{\partial q_i}). \qquad (i = 1,2,3) \tag{17}$$

$$\frac{\partial \overrightarrow{V}}{\partial \dot{q}_i} \;=\; \frac{\partial \vec{r}}{\partial q_i}. \qquad (i = 1,2,3) \tag{18}$$

$$\frac{d}{dt}(\frac{\partial \vec{r}}{\partial q_i}) \;=\; \frac{\partial^2 \vec{r}}{\partial q_i \partial q_1}\dot{q}_1 + \frac{\partial^2 \vec{r}}{\partial q_i \partial q_2}\dot{q}_2 + \frac{\partial^2 \vec{r}}{\partial q_i \partial q_3}\dot{q}_3. \qquad (i = 1,2,3) \tag{19}$$

$$\frac{\partial \overrightarrow{V}}{\partial q_i} \;=\; \frac{\partial^2 \vec{r}}{\partial q_1 \partial q_i}\dot{q}_1 + \frac{\partial^2 \vec{r}}{\partial q_2 \partial q_i}\dot{q}_2 + \frac{\partial^2 \vec{r}}{\partial q_3 \partial q_i}\dot{q}_3. \qquad (i = 1,2,3) \tag{20}$$

$$\frac{\partial \overrightarrow{V}}{\partial q_i} \;=\; \frac{d}{dt}(\frac{\partial \vec{r}}{\partial q_i}). \qquad (i = 1,2,3) \tag{21}$$

$$h_i a_{q_i} \;=\; \frac{d}{dt}(\overrightarrow{V} \cdot \frac{\partial \overrightarrow{V}}{\partial \dot{q}_i}) - \overrightarrow{V} \cdot \frac{\partial \overrightarrow{V}}{\partial q_i}. \qquad (i = 1,2,3) \tag{22}$$

$$\overrightarrow{V} \cdot \frac{\partial \overrightarrow{V}}{\partial \dot{q}_i} = \frac{\partial}{\partial \dot{q}_i}(\frac{V^2}{2}), \quad (i = 1, 2, 3) \tag{23}$$

$$\overrightarrow{V} \cdot \frac{\partial \overrightarrow{V}}{\partial q_i} = \frac{\partial}{\partial q_i}(\frac{V^2}{2}). \quad (i = 1, 2, 3) \tag{24}$$

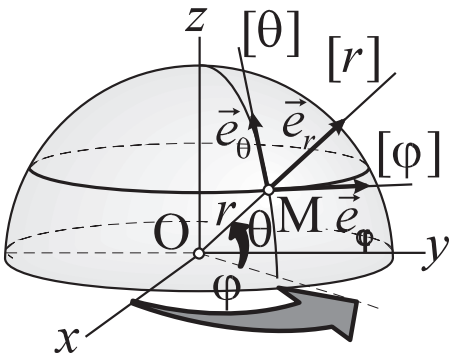
$$a_{q_i} = \frac{1}{h_i}\left\{\frac{d}{dt}[\frac{\partial}{\partial \dot{q}_i}(\frac{V^2}{2})] - \frac{\partial}{\partial q_i}(\frac{V^2}{2})\right\}, \qquad (i = 1, 2, 3) \tag{25}$$

$$\vec{a} = a_{q_1}\vec{e}_1 + a_{q_2}\vec{e}_2 + a_{q_3}\vec{e}_3 = \sum_{i=1}^{3} a_{q_i}\vec{e}_i. \tag{26}$$

$$a = \sqrt{a_{q_1}^2 + a_{q_2}^2 + a_{q_3}^2} = \sqrt{\sum_{i=1}^{3} a_{q_i}^2}, \tag{27}$$

при чему су величине a_{q_1}, a_{q_2} и a_{q_3} дате релацијама (25).

Брзина и убрзање тачке у сферним координатама



Слика 3

$$r = r(t), \quad \varphi = \varphi(t) \quad \text{и} \quad \theta = \theta(t).$$

$$x = r\cos\theta\cos\varphi, \qquad y = r\cos\theta\sin\varphi \quad \text{и} \quad z = r\sin\theta. \tag{28}$$

$$\begin{aligned}
\frac{\partial x}{\partial r} &= \cos\theta\cos\varphi, & \frac{\partial y}{\partial r} &= \cos\theta\sin\varphi, & \frac{\partial z}{\partial r} &= \sin\theta,\\
\frac{\partial x}{\partial \varphi} &= -r\cos\theta\sin\varphi, & \frac{\partial y}{\partial \varphi} &= r\cos\theta\cos\varphi, & \frac{\partial z}{\partial \varphi} &= 0,\\
\frac{\partial x}{\partial \theta} &= -r\sin\theta\cos\varphi, & \frac{\partial y}{\partial \theta} &= -r\sin\theta\sin\varphi, & \frac{\partial z}{\partial \theta} &= r\cos\theta,
\end{aligned} \tag{29}$$

Ламеови коефицијенти: $h_1 = 1$, $h_2 = r\cos\theta$ и $h_3 = r$,

$$\begin{aligned}
\vec{e}_1 &= \cos\theta\cos\varphi\vec{\imath} + \cos\theta\sin\varphi\vec{\jmath} + \sin\theta\vec{k},\\
\vec{e}_2 &= -\sin\varphi\vec{\imath} + \cos\varphi\vec{\jmath},\\
\vec{e}_3 &= -\sin\theta\cos\varphi\vec{\imath} - \sin\theta\sin\varphi\vec{\jmath} + \cos\theta\vec{k}.
\end{aligned} \tag{30}$$

$$V_r = \dot{r}, \qquad V_\varphi = r\dot{\varphi}\cos\theta \quad \text{и} \quad V_\theta = r\dot{\theta}, \tag{31}$$

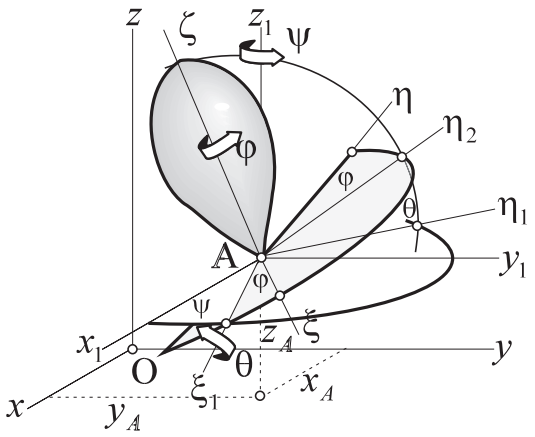
$$V = \sqrt{V_r^2 + V_\varphi^2 + V_\theta^2} = \sqrt{\dot{r}^2 + r^2\dot{\varphi}^2\cos^2\theta + r^2\dot{\theta}^2}. \tag{32}$$

$$\begin{aligned}
\frac{\partial}{\partial \dot{r}}\left(\frac{V^2}{2}\right) &= \dot{r}, & \frac{\partial}{\partial r}\left(\frac{V^2}{2}\right) &= r\dot{\varphi}^2\cos^2\theta + r\dot{\theta}^2,\\
\frac{\partial}{\partial \dot{\varphi}}\left(\frac{V^2}{2}\right) &= r^2\dot{\varphi}\cos^2\theta, & \frac{\partial}{\partial \varphi}\left(\frac{V^2}{2}\right) &= 0,\\
\frac{\partial}{\partial \dot{\theta}}\left(\frac{V^2}{2}\right) &= r^2\dot{\theta}, & \frac{\partial}{\partial \theta}\left(\frac{V^2}{2}\right) &= -r^2\dot{\varphi}^2\cos\theta\sin\theta,
\end{aligned} \tag{33}$$

$$\begin{aligned}
a_r &= \ddot{r} - r\dot{\varphi}^2\cos^2\theta - r\dot{\theta}^2,\\
a_\varphi &= \frac{1}{r\cos\theta}\frac{d}{dt}(r^2\dot{\varphi}\cos^2\theta) = r\ddot{\varphi}\cos\theta + 2\dot{r}\dot{\varphi}\cos\theta - 2r\dot{\varphi}\dot{\theta}\sin\theta,\\
a_\theta &= \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) + r\dot{\varphi}^2\sin\theta\cos\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} + r\dot{\varphi}^2\sin\theta\cos\theta,
\end{aligned} \tag{34}$$

$$a = \sqrt{a_r^2 + a_\varphi^2 + a_\theta^2} \tag{35}$$

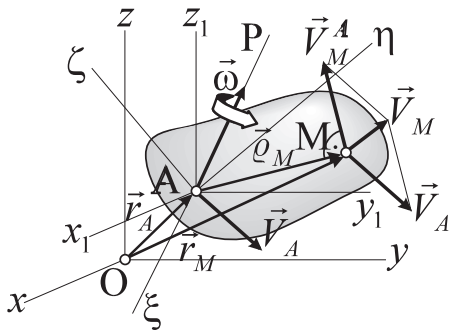
Опште кретање слободног крутог тела



Слика 4

$$\begin{aligned} x_A &= x_A(t), \quad y_A = y_A(t), \quad z_A = z_A(t), \\ \psi &= \psi(t), \qquad \theta = \theta(t), \qquad \varphi = \varphi(t), \end{aligned} \tag{36}$$

Брзине тачака слободног крутог тела при општем кретању



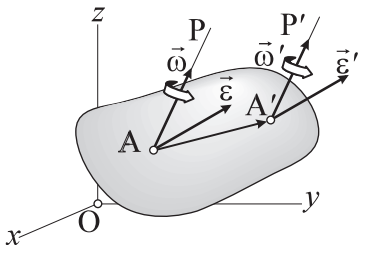
Слика 5

$$\vec{r}_M = \vec{r}_A + \vec{\varrho}_M, \tag{37}$$

$$\overrightarrow{V}_M = \frac{d\vec{r}_A}{dt} + \frac{d\vec{\varrho}_M}{dt}, \tag{38}$$

$$\frac{d\vec{\varrho}_M}{dt} = \vec{\omega} \times \vec{\varrho}_M,$$

$$\overrightarrow{V}_M = \overrightarrow{V}_A + \vec{\omega} \times \vec{\varrho}_M = \overrightarrow{V}_A + \overrightarrow{V}_M^A, \tag{39}$$



$$\overrightarrow{V}_{A'} = \overrightarrow{V}_A + \vec{\omega} \times \overrightarrow{AA'}, \tag{40}$$

$$\overrightarrow{V}_A = \overrightarrow{V}_{A'} + \vec{\omega}' \times \overrightarrow{A'A} = \overrightarrow{V}_{A'} - \vec{\omega}' \times \overrightarrow{AA'}, \tag{41}$$

$$(\vec{\omega} - \vec{\omega}') \times \overrightarrow{AA'} = 0. \tag{42}$$

Слика 6

$$\vec{\omega} - \vec{\omega}' = 0, \qquad \text{односно} \qquad \vec{\omega} = \vec{\omega}', \tag{43}$$

$$\frac{d\vec{\omega}}{dt} = \frac{d\vec{\omega}'}{dt}, \qquad \text{односно} \qquad \vec{\varepsilon} = \vec{\varepsilon}', \tag{44}$$

Убрзања тачака слободног крутог тела при општем кретању

$$\begin{aligned} \vec{a}_M = \frac{d\overrightarrow{V}_M}{dt} &= \frac{d\overrightarrow{V}_A}{dt} + \frac{d}{dt}(\vec{\omega} \times \vec{\varrho}_M) = \\ &= \frac{d\overrightarrow{V}_A}{dt} + \frac{d\vec{\omega}}{dt} \times \vec{\varrho}_M + \vec{\omega} \times \frac{d\vec{\varrho}_M}{dt} = \\ &= \vec{a}_A + \vec{\varepsilon} \times \vec{\varrho}_M + \vec{\omega} \times \overrightarrow{V}_M^A, \end{aligned} \tag{45}$$

$$\vec{a}_M^A = \vec{\varepsilon} \times \vec{\varrho}_M + \vec{\omega} \times \overrightarrow{V}_M^A, \tag{46}$$

$$\vec{a}_M = \vec{a}_A + \vec{a}_M^A, \tag{47}$$