Tablica integrala

1. $\int x^{\alpha} dx = \frac{x^{\alpha+1}}{\alpha+1} + C, \quad \alpha \neq -1.$

2. $\int \frac{dx}{x+a} = \log|x+a| + C; \quad \int \frac{dx}{x} = \log|x| + C.$

3. $\int a^x dx = \frac{a^x}{\log a} + C, \quad a > 0, \ a \neq 1.$

4. $\int e^x dx = e^x + C.$

5. $\int \sin x \, dx = -\cos x + C.$

6. $\int \cos x \, dx = \sin x + C.$

7. $\int \frac{dx}{\cos^2 x} = \operatorname{tg} x + C.$

8. $\int \frac{dx}{\sin^2 x} = -\operatorname{ctg} x + C.$

9. $\int \operatorname{sh} x \, dx = \operatorname{ch} x + C.$

10. $\int \operatorname{ch} x \, dx = \operatorname{sh} x + C.$

11. $\int \frac{dx}{\operatorname{ch}^2 x} = \operatorname{th} x + C$

12. $\int \frac{dx}{\operatorname{sh}^2 x} = -\operatorname{cth} x + C.$

13. $\int \operatorname{tg} x \, dx = -\log|\cos x| + C.$

14. $\int ctg\,x\,dx = \log|\sin x| + C.$

15. $\int \frac{dx}{\sin x} = \log\left|tg\frac{x}{2}\right| + C.$

16. $\int \frac{dx}{x^2+1} = arctg\,x + C.$

17. $\int \frac{dx}{x^2-1} = \frac{1}{2}\log\left|\frac{x-1}{x+1}\right| + C.$

18. $\int \frac{dx}{\sqrt{1-x^2}} = \arcsin x + C =$

$= -\arccos x + C_1$

19. $\int \frac{dx}{\sqrt{x^2+1}} = \log(x+\sqrt{x^2+1}) + C.$

20. $\int \frac{dx}{\sqrt{x^2-1}} = \log|x+\sqrt{x^2-1}| + C.$

21. $\int \frac{f'(x)}{f(x)}\,dx = \log|f(x)| + C.$

22. $\int \frac{f'(x)}{f^2(x)}\,dx = -\frac{1}{f(x)} + C.$

23. $\int \frac{f'(x)}{\sqrt{f(x)}}\,dx = 2\sqrt{f(x)} + C.$

24. $\int f'(x)\,e^{f(x)}\,dx = e^{f(x)} + C.$

25. $\int x^x(\log x+1)\,dx = x^x + C.$

26. $\left(\int f(x)dx\right)' = f(x)$

27. $d\left(\int f(x)dx\right) = f(x)\,dx.$

28. $\int f'(x)\,dx = f(x) + C$

29. $\int df(x) = f(x) + C$

30. $\int a f(x)\,dx = a\int f(x)\,dx$

(a - konstanta)

31. $\int (f_1(x)+f_2(x))\,dx =$

$= \int f_1(x)\,dx + \int f_2(x)\,dx$

$\int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin\frac{x}{a} + C$

$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\,arctg\,\frac{x}{a} + C$

$\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C$

$\int \frac{dx}{\sqrt{x^2\pm a^2}} = \ln(x+\sqrt{x^2\pm a^2}) + C$

СКОРО ТАБЛИЧНИ ИНТЕГРАЛИ

1. $\int \left(\frac{1}{\sqrt{x}} + \frac{1}{3}x - 3\sqrt{x}\right)dx = \int x^{-1/2}dx + \frac{1}{3}\int x\,dx - 3\int x^{1/2}dx$

$= \frac{x^{1/2}}{1/2} + \frac{1}{3}\cdot\frac{x^2}{2} - 3\cdot\frac{x^{3/2}}{3/2} + c$

$= 2\sqrt{x} + \frac{x^2}{6} - 2\sqrt{x^3} + c$

2. $\int \frac{(x-1)^2}{\sqrt{x}}dx = \int \frac{x^2-2x+1}{\sqrt{x}}dx = \int x^{3/2}dx - 2\int \sqrt{x}\,dx + \int x^{-1/2}dx$

$= \frac{2}{5}x^{5/2} - \frac{4}{3}x^{3/2} + 2x^{1/2} + c$

3. $\int \sqrt{x\sqrt{x\sqrt{x}}}\,dx = \int x^{1/2}x^{1/4}x^{1/8}dx = \int x^{7/8}dx$

$= \frac{8}{15}x^{15/8} + c$

4. $\int \frac{dx}{x^2(1+x^2)} = \int \frac{1+x^2-x^2}{x^2(1+x^2)}dx = \int \frac{dx}{x^2} - \int \frac{dx}{1+x^2}$

$= -\frac{1}{x} - \operatorname{arctg} x + c$

5. $\int \cos^2\frac{x}{2}dx = \int \frac{1+\cos x}{2}dx = \frac{1}{2}\int dx + \frac{1}{2}\int \cos x\,dx$

$= \frac{x}{2} + \frac{\sin x}{2} + c$

6. $\int \operatorname{tg}^2 x\,dx = \int \frac{\sin^2 x}{\cos^2 x}dx = \int \frac{1-\cos^2 x}{\cos^2 x}dx$

$= \int \frac{1}{\cos^2 x}dx - \int dx = \operatorname{tg} x - x + c$

7. $\int 3^x\cdot 5^{2x}dx = \int (3\cdot 5^2)^x dx = \int 75^x dx = \frac{75^x}{\log 75} + c$

8. $\int \frac{2^x+5^x}{10^x}dx = \int \left(\frac{1}{5}\right)^x dx + \int \left(\frac{1}{2}\right)^x dx$

$= \frac{\left(\frac{1}{5}\right)^x}{\log\frac{1}{5}} + \frac{\left(\frac{1}{2}\right)^x}{\log\frac{1}{2}} + c$

ИНТЕГРАЦИЈА МЕТОДОМ СМЕНЕ

$$\int f(\varphi(x))\varphi'(x)dx = \left| \begin{array}{l} \varphi(x) = t \\ \varphi'(x)dx = dt \end{array} \right| = \int f(t)dt$$

1. a) $$\int (x+1)^{15}dx = \left| \begin{array}{l} x+1 = t \\ dx = dt \end{array} \right| = \int t^{15}dt = \frac{t^{16}}{16} + c = \frac{(x+1)^{16}}{16} + c$$

б) $$\int \frac{dx}{(2x-3)^5} = \left| \begin{array}{l} 2x-3 = t \\ 2dx = dt \\ dx = \frac{dt}{2} \end{array} \right| = \frac{1}{2}\int \frac{dt}{\cancel{\sqrt{}}t^5} = \frac{1}{2}\,\frac{t^{-4}}{-4} = -\frac{1}{8}(2x-3)^{-4} + c$$

c) $$\int \sqrt{8-2x}\,dx = \left| \begin{array}{l} 8-2x = t \\ -2dx = dt \\ dx = \frac{dt}{-2} \end{array} \right| = -\frac{1}{2}\int \sqrt{t}\,dt = -\frac{1}{2}\int t^{1/2}dt$$

$$= -\frac{1}{2}\,\frac{t^{3/2}}{3/2} + c = -\frac{1}{3}(8-2x)^{3/2} + c$$

2. a) $$\int x\sqrt{x^2+1}\,dx = \left| \begin{array}{l} x^2+1 = t \\ 2x\,dx = dt \\ x\,dx = \frac{dt}{2} \end{array} \right| = \frac{1}{2}\int \sqrt{t}\,dt$$

$$= \frac{1}{3}t^{3/2} + c = \frac{1}{3}(x^2+1)^{3/2} + c$$

б) $$\int \frac{x^4}{\sqrt{4+x^5}}dx = \left| \begin{array}{l} 4+x^5 = t \\ 5x^4dx = dt \\ x^4dx = \frac{dt}{5} \end{array} \right| = \frac{1}{5}\int \frac{dt}{\sqrt{t}} = \frac{1}{5}\int t^{-1/2}dt$$

$$= \frac{1}{5}\,\frac{t^{1/2}}{1/2} + c = \frac{2}{5}\sqrt{t} + c = \frac{2}{5}\sqrt{4+x^5} + c$$

c) $$\int \frac{x\,dx}{x^4+1} = \left| \begin{array}{ll} x^2 = t & x^4+1 = t^2+1 \\ 2x\,dx = dt & \\ x\,dx = \frac{dt}{2} & \end{array} \right|$$

$$= \frac{1}{2}\int \frac{dt}{t^2+1} = \frac{1}{2}\operatorname{arctg} t + c = \frac{1}{2}\operatorname{arctg} x^2 + c$$

3. a) $\int \frac{e^x}{e^x+1}\,dx = \left| \begin{matrix} e^x+1=t \\ e^x dx = dt \end{matrix} \right| = \int \frac{dt}{t} = \log|t| + c$

$= \log(e^x+1) + c$

b) $\int e^x \sqrt{e^x+1}\,dx = \left| \begin{matrix} e^x+1=t \\ e^x dx = dt \end{matrix} \right| = \int \sqrt{t}\,dt = \frac{2}{3} t^{3/2} + c$

$= \frac{2}{3}\sqrt{(e^x+1)^3} + c$

c) $\int \frac{dx}{e^x+e^{-x}} = \int \frac{e^x}{e^{2x}+1}\,dx = \left| \begin{matrix} e^x = t \\ e^x dx = dt \end{matrix} \right|$

$= \int \frac{dt}{t^2+1} = \operatorname{arctg} t + c = \operatorname{arctg} e^x + c$

4. a) $\int \frac{\sqrt{\log x}}{x}\,dx = \left| \begin{matrix} \log x = t \\ \frac{1}{x}dx = dt \end{matrix} \right| = \int \sqrt{t}\,dt = \frac{2}{3} t^{3/2} + c$

$= \frac{2}{3}(\log x)^{3/2} + c$

b) $\int \frac{dx}{x\log x} = \left| \begin{matrix} \log x = t \\ \frac{1}{x}dx = dt \end{matrix} \right| = \int \frac{dt}{t} = \log|t| + c$

$= \log|\log x| + c$

c) $\int \frac{dx}{x \log x \log\log x} = \left| \begin{matrix} \log x = t \\ \frac{dx}{x} = dt \end{matrix} \right| = \int \frac{dt}{t\cdot \log t} \overset{4\,b)}{=}$

$= \log|\log t| + c = \log|\log\log x| + c$

5. a) $\int (\sin 2x + \cos 3x)\,dx = \int \sin 2x\,dx + \int \cos 3x\,dx$

$\left| \begin{matrix} 2x = u \\ 2dx = du \\ dx = \frac{du}{2} \end{matrix} \right| \qquad \left| \begin{matrix} 3x = v \\ dx = \frac{dv}{3} \end{matrix} \right|$

$= \frac{1}{2}\int \sin u\,du + \frac{1}{3}\int \cos v\,dv$

$= -\frac{1}{2}\cos u + \frac{1}{3}\sin v + c$

$= -\frac{1}{2}\cos 2x + \frac{1}{3}\sin 3x + c$

б) $\int \sin^3 x \cos x \, dx = \left| \begin{array}{l} \sin x = t \\ \cos x \, dx = dt \end{array} \right| = \int t^3 dt = \frac{1}{4} t^4 + c$

$= \frac{1}{4} \sin^4 x + c$

c) $\int \frac{\sin x}{\cos^2 x} dx = \left| \begin{array}{l} \cos x = t \\ -\sin x \, dx = dt \end{array} \right| = -\int \frac{dt}{t^2} = \frac{1}{t} + c = \frac{1}{\cos x} + c$

d) $\int \frac{dx}{2+\cos^2 x} = \int \frac{dx}{3\cos^2 x + 2\sin^2 x} \; /:\cos^2 x = \int \frac{\frac{dx}{\cos^2 x}}{3+2tg^2 x} = \left| \begin{array}{l} tg\, x = t \\ \frac{dx}{\cos^2 x} = dt \end{array} \right|$

$= \int \frac{dx}{3+2t^2} = \frac{1}{2} \int \frac{dx}{t^2+\frac{3}{2}} = \frac{1}{2} \frac{1}{\sqrt{\frac{3}{2}}} \operatorname{arctg} \frac{t}{\sqrt{\frac{3}{2}}} + c$

$= \frac{1}{\sqrt{6}} \operatorname{arctg} \frac{\sqrt{2}\, tg\, x}{\sqrt{3}} + c$

6. a) $\int \frac{dx}{2+\sqrt{x}} = \left| \begin{array}{l} \sqrt{x} = t \\ \frac{dx}{2\sqrt{x}} = dt \\ dx = 2\sqrt{x}\, dt \\ dx = 2t\, dt \end{array} \right|$ или $\left| \begin{array}{l} \sqrt{x} = t \\ x = t^2 \\ dx = 2t\, dt \end{array} \right|$

$= \int \frac{2t\,dt}{2+t} = 2\int \frac{t}{2+t} dt = 2\int \frac{t+2-2}{2+t} dt$

$= 2\int dt - 4\int \frac{dt}{2+t} = 2t - 4\log|2+t| + c$

$= 2\sqrt{x} - 4\log(2+\sqrt{x}) + c$

б) $\int \frac{x^3}{\sqrt{x-1}} dx = \left| \begin{array}{l} \sqrt{x-1} = t \\ x-1 = t^2 \\ x = t^2+1 \\ dx = 2t\,dt \end{array} \right| = \int \frac{(t^2+1)^3}{t} \cdot 2t\,dt = 2\int (t^2+1)^3 dt$

$= 2\int (t^6 + 3t^4 + 3t^2 + 1)\, dt$

$= 2\frac{t^7}{7} + 6\frac{t^5}{5} + 6\frac{t^3}{3} + 2t + c$

$= \frac{2}{7} (x-1)^{7/2} + \frac{6}{5}(x-1)^{5/2} + 2(x-1)^{3/2} + 2(x-1)^{1/2} + c$

EO c) $\int \frac{\sqrt{x}\,dx}{\sqrt[3]{x}-1} = \left| \begin{array}{l} \sqrt[6]{x} = t \\ x = t^6 \\ dx = 6t^5 dt \end{array} \right| = \int \frac{t^3}{t^2-1} \cdot 6t^5 dt$

7. $\displaystyle\int \frac{dx}{ax^2+bx+c}$

а) $\displaystyle I=\int \frac{dx}{x^2+4x+3}$

Ако је квадратни трином растављив на чиниоце, подинтегрална функција се раставља на просте разломке

$$x^2+4x+3=(x+3)(x+1)$$

$$\frac{1}{(x+3)(x+1)}=\frac{A}{x+3}+\frac{B}{x+1}=\frac{(A+B)x+(A+3B)}{(x+3)(x+1)}$$

$$\Rightarrow \left.\begin{array}{l} A+B=0 \\ A+3B=1 \end{array}\right\} \Rightarrow A=-\frac{1}{2}\ ,\ B=\frac{1}{2}$$

$$I=-\frac{1}{2}\int\frac{dx}{x+3}+\frac{1}{2}\int\frac{dx}{x+1}=-\frac{1}{2}\log|x+3|+\frac{1}{2}\log|x+1|+c$$

$$=\frac{1}{2}\log\left|\frac{x+1}{x+3}\right|+c$$

б) $\displaystyle I=\int\frac{dx}{x^2+2x+5}$

Квадратни трином у имениоцу нема реалних нула, па се трансформише на облик потпуног квадрата:

$$x^2+2x+5=x^2+2x+1+4=(x+1)^2+4$$

$$I=\int\frac{dx}{(x+1)^2+4}=\frac{1}{4}\int\frac{dx}{\left(\frac{x+1}{2}\right)^2+1}=\left|\begin{array}{l}\frac{x+1}{2}=t\\ dx=2dt\end{array}\right|$$

$$=\frac{1}{4}\int\frac{2dt}{t^2+1}=\frac{1}{2}\operatorname{arctg} t+c=\frac{1}{2}\operatorname{arctg}\frac{x+1}{2}+c$$

Д ⊛ $\displaystyle\int\frac{dx}{x(x-1)}$, $\displaystyle\int\frac{dx}{x^2-7x+10}$

$\displaystyle\int\frac{dx}{x^2+10x+31}$, $\displaystyle\int\frac{dx}{25-8x+x^2}$, $\displaystyle\int\frac{dx}{2x^2-2x+5}$

8. $\int \frac{Ax+B}{ax^2+bx+c}\,dx$

a) $\int \frac{2x-2}{x^2+6x+13}\,dx = \int \frac{2x+6}{x^2+6x+13}\,dx - \int \frac{8}{x^2+6x+13}\,dx$

$$\left[\begin{array}{l} x^2+6x+13 = t \\ (2x+6)\,dx = dt \end{array}\right|$$

$$= \int \frac{dt}{t} - 8 \int \frac{dx}{(x+3)^2+4}$$

$$= \log|t| - 2 \int \frac{dx}{\left(\frac{x+3}{2}\right)^2+1} = \left| \begin{array}{l} \frac{x+3}{2} = u \\ dx = 2du \end{array} \right|$$

$$= \log(x^2+6x+13) - 2\int \frac{2du}{u^2+1}$$

$$= \log(x^2+6x+13) - 4\,\mathrm{arctg}\,\frac{x+3}{2} + c$$

b) $\int \frac{3x-1}{x^2-x+1}\,dx = \frac{3}{2}\int \frac{2x-\frac{2}{3}}{x^2-x+1}\,dx$

$$= \frac{3}{2}\int \frac{2x-1}{x^2-x+1}\,dx + \frac{3}{2}\int \frac{\frac{1}{3}}{x^2-x+1}\,dx$$

$$= \frac{3}{2}\log(x^2-x+1) + \frac{1}{2}\int \frac{dx}{x^2-x+1}$$

$$\left[x^2-x+1 = \left(x-\tfrac{1}{2}\right)^2+\tfrac{3}{4} \right]$$

$$= \frac{3}{2}\log(x^2-x+1) + \frac{1}{2}\int \frac{dx}{\left(x-\frac{1}{2}\right)^2+\frac{3}{4}}$$

$$= \frac{3}{2}\log(x^2-x+1) + \frac{1}{2}\cdot\frac{4}{3}\int \frac{dx}{\left(\frac{x-\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)^2+1}$$

$$= \frac{3}{2}\log(x^2-x+1) + \frac{2}{3}\int \frac{dx}{\left(\frac{2x-1}{\sqrt{3}}\right)^2+1} = \left| \begin{array}{l} \frac{2x-1}{\sqrt{3}} = t \\ dx = \frac{\sqrt{3}}{2}\,dt \end{array} \right|$$

$$= \frac{3}{2}\log(x^2-x+1) + \frac{2}{3}\cdot\frac{\sqrt{3}}{2}\int \frac{dt}{t^2+1}$$

$$= \frac{3}{2}\log(x^2-x+1) + \frac{1}{\sqrt{3}}\,\mathrm{arctg}\,\frac{2x-1}{\sqrt{3}} + c$$

9. $\displaystyle\int \frac{dx}{\sqrt{ax^2+bx+c}}$ и $\displaystyle\int \frac{Ax+B}{\sqrt{ax^2+bx+c}}\,dx$

a) $$\int \frac{dx}{\sqrt{x+x^2}} = \int \frac{dx}{\sqrt{(x+\frac{1}{2})^2-\frac{1}{4}}} = \left| \begin{matrix} x+\frac{1}{2}=t \\ dx=dt \end{matrix} \right| = \int \frac{dt}{\sqrt{t^2-\frac{1}{4}}}$$

$$= \log\left| t+\sqrt{t^2-\tfrac{1}{4}} \right| + c$$

$$= \log\left| (x+\tfrac{1}{2}) + \sqrt{(x+1/2)^2-1/4} \right| + c$$

$$= \log\left| x+\tfrac{1}{2}+\sqrt{x^2+x} \right| + c$$

б) $$\int \frac{dx}{\sqrt{x^2-2x+5}} = \int \frac{dx}{\sqrt{(x-1)^2+4}} = \left| \begin{matrix} x-1=t \\ dx=dt \end{matrix} \right| = \int \frac{dt}{\sqrt{t^2+4}}$$

$$= \log\left| t+\sqrt{t^2+4} \right| + c = \log\left| x-1+\sqrt{x^2-2x+5} \right| + c$$

c) $$\int \frac{dx}{\sqrt{5-4x-x^2}} = \int \frac{dx}{\sqrt{9-(x+2)^2}} = \left| \begin{matrix} x+2=t \\ dx=dt \end{matrix} \right|$$

$$= \int \frac{dt}{\sqrt{9-t^2}} = \arcsin\frac{t}{3} + c = \arcsin\frac{x+2}{3} + c$$

d) $$\int \frac{x+3}{\sqrt{27+6x-x^2}}\,dx = -\frac{1}{2}\int \frac{-2x-6}{\sqrt{27+6x-x^2}}\,dx$$

$$= -\frac{1}{2}\int \frac{-2x+6}{\sqrt{27+6x-x^2}}\,dx + \frac{1}{2}\int \frac{-12}{\sqrt{27+6x-x^2}}\,dx$$

$$= -\frac{1}{2}\int \frac{dt}{\sqrt{t}} + 6\int \frac{dx}{\sqrt{36-(x-3)^2}}$$

$$= -\sqrt{27+6x-x^2} + 6\arcsin\frac{x-3}{6} + c$$

e) $$\int \frac{x+5}{\sqrt{x^2+6x-27}}\,dx = \frac{1}{2}\int \frac{2x+6}{\sqrt{x^2+6x-27}} + 2\int \frac{dx}{\sqrt{(x+3)^2-36}}$$

$$= \sqrt{x^2+6x-27} + 2\log\left| x+3+\sqrt{(x+3)^2-36} \right| + c$$

ПАРЦИЈАЛНА ИНТЕГРАЦИЈА ([redacted])

$$\int u\,dv = uv - \int v\,du$$

1. a) $$\int x e^x dx = \left|\begin{matrix} x = u & e^x dx = dv \\ du = dx & v = e^x \end{matrix}\right| = xe^x - \int e^x dx = xe^x - e^x + c$$

   б) $$\int x^3 e^{2x} dx = \left|\begin{matrix} x^3 = u & e^{2x}dx = dv \\ du = 3x^2 & v = \int e^{2x}dx \\ & v = \frac{1}{2}e^{2x} \end{matrix}\right| = \frac{1}{2}x^3 e^{2x} - \frac{3}{2}\int x^2 e^{2x} dx$$

$$= \left|\begin{matrix} x^2 = u & e^{2x}dx = dv \\ du = 2x & v = \frac{1}{2}e^{2x} \end{matrix}\right| = \frac{1}{2}x^3 e^{2x} - \frac{3}{2}\left(\frac{1}{2}x^2 e^{2x} - \frac{2}{2}\int x e^{2x}\right.$$

$$= \left|\begin{matrix} x = u & e^{2x}dx = dv \\ du = dx & v = \frac{1}{2}e^{2x} \end{matrix}\right|$$

$$= \frac{1}{2}x^3 e^{2x} - \frac{3}{4}x^2 e^{2x} + \frac{3}{4}x e^{2x} - \frac{3}{8}e^{2x} + c$$

У општем случају: $\int P_n(x) e^{ax} dx = \left|\begin{matrix} P_n(x) = u \\ e^{ax}dx = dv \end{matrix}\right|$

2. $$\int x \sin x\, dx = \left|\begin{matrix} x = u & \sin x\,dx = dv \\ du = dx & v = \int \sin x\,dx \\ & v = -\cos x \end{matrix}\right| = -x\cos x + \int \cos x\,dx$$

$$= -x\cos x + \sin x + c.$$

У општем случају: $\int P_n(x)\sin ax\,dx = \left|\begin{matrix} P_n(x) = u \\ \sin ax\,dx = dV \end{matrix}\right|$

$\int P_n(x)\cos ax\,dx = \left|\begin{matrix} P_n(x) = u \\ \cos ax\,dx = dV \end{matrix}\right|$

3. a) $$\int \arcsin x\, dx = \left|\begin{matrix} u = \arcsin x & dx = dv \\ du = \frac{dx}{\sqrt{1-x^2}} & v = x \end{matrix}\right|$$

$$= x\arcsin x - \int \frac{x}{\sqrt{1-x^2}}dx = \left|\begin{matrix} 1-x^2 = t \\ -2x\,dx = dt \\ x\,dx = -\frac{dt}{2} \end{matrix}\right|$$

$$= x\arcsin x + \frac{1}{2}\int\frac{dt}{\sqrt{t}} = x\arcsin x + \sqrt{t} + c$$

$$= x\arcsin x + \sqrt{1-x^2} + c$$

b) $\int x \operatorname{arctg} x \, dx = \left| \begin{array}{ll} u = \operatorname{arctg} x & x\,dx = dv \\ du = \frac{dx}{1+x^2} & v = \frac{x^2}{2} \end{array} \right|$

$$= \frac{x^2}{2} \operatorname{arctg} x - \frac{1}{2} \int \frac{x^2}{1+x^2}\, dx$$

$$= \frac{x^2}{2} \operatorname{arctg} x - \frac{1}{2} \int \frac{x^2+1-1}{1+x^2}\, dx$$

$$= \frac{x^2}{2} \operatorname{arctg} x - \frac{1}{2} \int dx + \frac{1}{2} \int \frac{dx}{1+x^2}$$

$$= \frac{x^2}{2} \operatorname{arctg} x - \frac{1}{2} x + \frac{1}{2} \operatorname{arctg} x + c$$

$$= \frac{x^2+1}{2} \operatorname{arctg} x - \frac{1}{2} x + c.$$

У општем случају: $\int P_n(x) \operatorname{arctg} x \, dx = \left| \begin{array}{l} \operatorname{arctg} x = u \\ P_n(x)\,dx = dV \end{array} \right|$

$\int P_n(x) \arcsin x \, dx = \left| \begin{array}{l} \arcsin x = u \\ P_n(x)\,dx = dV \end{array} \right|$

4. a) $\int \ln x \, dx = \left| \begin{array}{ll} \ln x = u & dx = dV \\ du = \frac{dx}{x} & v = x \end{array} \right| = x \ln x - \int x \frac{dx}{x} =$

$$= x \ln x - x + c$$

b) $\int \ln(x^2+1)\, dx = \left| \begin{array}{ll} \ln(x^2+1) = u & dx = dv \\ du = \frac{2x\,dx}{1+x^2} & v = x \end{array} \right|$

$$= x \ln(x^2+1) - 2 \int \frac{x^2 \pm 1}{1+x^2}\, dx$$

$$= x \ln(x^2+1) - 2 \int dx + 2 \int \frac{dx}{1+x^2}$$

$$= x \ln(x^2+1) - 2x + 2 \operatorname{arctg} x + c.$$

c) $\int \ln\left(x + \sqrt{4+x^2}\right) dx = \left| \begin{array}{ll} \ln\left(x+\sqrt{4+x^2}\right) = u & dx = dv \\ du = \frac{1}{x+\sqrt{4+x^2}} \cdot \left(1 + \frac{2x}{2\sqrt{4+x^2}}\right) dx & v = x \\ du = \frac{1}{x+\sqrt{4+x^2}} \cdot \frac{\sqrt{4+x^2}+x}{\sqrt{4+x^2}}\, dx & \\ du = \frac{dx}{\sqrt{4+x^2}} & \end{array} \right|$

5. a) $I=\int e^{x}\sin x\,dx = \begin{vmatrix} u=\sin x & e^{x}dx=dv \\ du=\cos x\,dx & v=e^{x}\end{vmatrix}$

$= e^{x}\sin x - \int e^{x}\cos x\,dx = \begin{vmatrix} u=\cos x & e^{x}dx=dv \\ du=-\sin x\,dx & v=e^{x}\end{vmatrix}$

$= e^{x}\sin x - e^{x}\cos x - \underbrace{\int e^{x}\sin x\,dx}_{I}$

$2I = e^{x}\sin x - e^{x}\cos x$

$I = \frac{e^{x}}{2}(\sin x-\cos x)+c$

b) $A=\int e^{ax}\cos bx\,dx$, $B=\int e^{ax}\sin bx\,dx$, $a^2+b^2\neq 0$

$A=\begin{vmatrix} u=e^{ax} & \cos bx\,dx=dv \\ du=ae^{ax}dx & v=\frac{1}{b}\sin bx\end{vmatrix} = \frac{1}{b}e^{ax}\sin bx - \frac{a}{b}\int e^{ax}\sin bx$

$\underset{(*)}{=} \frac{1}{b}e^{ax}\sin bx - \frac{a}{b}\cdot B = \begin{vmatrix} u=e^{ax} & \sin bx\,dx=dv \\ du=ae^{ax} & v=-\frac{1}{b}\cos bx\end{vmatrix}$

$= \frac{1}{b}e^{ax}\sin bx - \frac{a}{b}\cdot\left(-\frac{1}{b}e^{ax}\cos bx + \frac{a}{b}\int e^{ax}\cos bx\,dx\right)$

$= \frac{1}{b}e^{ax}\sin bx + \frac{a}{b^2}e^{ax}\cos bx - \frac{a^2}{b^2}\cdot A$

$\Rightarrow \quad \frac{a^2+b^2}{b^2}\cdot A = \frac{e^{ax}}{b^2}\left(b\sin bx + a\cos bx\right)$

$A=\frac{e^{ax}}{a^2+b^2}\left(b\sin bx + a\cos bx\right)+c$

$\overset{*}{\Rightarrow} \quad B=\frac{e^{ax}}{a^2+b^2}\cdot\left(a\sin bx - b\cos bx\right)+c$

A ⊛ $\int e^{2x}\cos 3x\,dx$

6. $I = \int \sqrt{a^2 - x^2}\, dx \quad , \ a > 0$

$$I = \left| \begin{array}{ll} \sqrt{a^2-x^2} = u & dx = dv \\ du = -\dfrac{x}{\sqrt{a^2-x^2}}\, dx & v = x \end{array} \right| = x\sqrt{a^2-x^2} + \int \frac{x^2}{\sqrt{a^2-x^2}}\, dx$$

$$= x\sqrt{a^2-x^2} + \int \frac{x^2 - a^2 + a^2}{\sqrt{a^2-x^2}}\, dx =$$

$$= x\sqrt{a^2-x^2} - \int \frac{a^2-x^2}{\sqrt{a^2-x^2}}\, dx + \int \frac{a^2}{\sqrt{a^2-x^2}}\, dx$$

$$= x\sqrt{a^2-x^2} - \underbrace{\int \sqrt{a^2-x^2}\, dx}_{I} + a^2 \int \frac{dx}{\sqrt{a^2-x^2}}$$

$$= x\sqrt{a^2-x^2} - I + a^2 \arcsin\frac{x}{a} + c_1$$

$$\Rightarrow 2I = x\sqrt{a^2-x^2} + a^2 \arcsin\frac{x}{a} + c_1$$

$$I = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\arcsin\frac{x}{a} + c$$

b) $I = \int \sqrt{a^2+x^2}\, dx = \left| \begin{array}{ll} \sqrt{a^2+x^2} = u & dx = dv \\ du = \dfrac{x\,dx}{\sqrt{a^2+x^2}} & v = x \end{array} \right|$

$$= x\sqrt{a^2+x^2} - \int \frac{x^2 \pm a^2}{\sqrt{a^2+x^2}}\, dx =$$

$$= x\sqrt{a^2+x^2} - \underbrace{\int \sqrt{a^2+x^2}\, dx}_{I} + a^2 \int \frac{dx}{\sqrt{a^2+x^2}}$$

$$= x\sqrt{a^2+x^2} - I + a^2 \ln\left(x + \sqrt{a^2+x^2}\right) + c_1$$

$$\Rightarrow I = \frac{x}{2}\sqrt{a^2+x^2} + \frac{a^2}{2}\ln\left(x + \sqrt{a^2+x^2}\right) + c$$

ИНТЕГРАЦИЈА РАЦИОНАЛНИХ Ф-ЈА (~~ТИ, ЧМ~~)

$$R(x) = \frac{P(x)}{Q(x)}$$

1. Ако је $d^\circ P \geq d^\circ Q$ изврши се дељење полинома $P(x)$ са $Q(x)$

$$\frac{P(x)}{Q(x)} = q(x) + \frac{r(x)}{Q(x)}, \quad \text{где је } d^\circ r < d^\circ Q$$

2. Ако је $d^\circ P < d^\circ Q$ полином $Q(x)$ се раставља на просте чиниоце, па се $R(x)$ представља у облику:

$$\frac{P(x)}{Q(x)} = \frac{P(x)}{(x-\alpha_1)^{m_1}\cdots(x-\alpha_i)^{m_i}\,(a_1x^2+b_1x+c_1)^{m_1}\cdots(a_jx^2+b_jx+c_j)^{m_j}}$$

$$= \frac{A_1}{(x-\alpha_1)^{m_1}} + \cdots + \frac{A_i}{(x-\alpha_i)^{m_i}} + \frac{B_1x+C_1}{(a_1x^2+b_1x+c_1)^{m_1}} + \cdots + \frac{B_jx+C_j}{(a_jx^2+b_jx+c_j)^{m_j}}$$

Овим се $\int \frac{P(x)}{Q(x)}dx$, где је $d^\circ P < d^\circ Q$, своди на израчунавање нама познатих интеграла

$$\int \frac{dx}{x-\alpha}, \quad \int \frac{dx}{(x-\alpha)^m}, \quad \int \frac{dx}{ax^2+bx+c} \quad \text{и} \quad \int \frac{dx}{(ax^2+bx+c)^m}$$

1. $\displaystyle\int \frac{dx}{(x-1)(x-2)(x-3)}$

I НАЧИН:

$$\frac{1}{(x-1)(x-2)(x-3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3} = \frac{A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2)}{(x-1)(x-2)(x-3)}$$

$$1 = A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2)$$

за $x = 1$: $1 = +2A \Rightarrow A = \frac{1}{2}$

$x = 2$: $1 = -B \Rightarrow B = -1$

$x = 3$: $1 = 2C \Rightarrow C = \frac{1}{2}$

Дакле,

$$\int \frac{dx}{(x-1)(x-2)(x-3)} = \frac{1}{2}\int\frac{dx}{x-1} - \int\frac{dx}{x-2} + \frac{1}{2}\int\frac{dx}{x-3}$$

$$= \frac{1}{2}\log|x-1| - \log|x-2| + \frac{1}{2}\log|x-3| + C$$

II НАЧИН:

$$\frac{1}{(x-1)(x-2)(x-3)} = \frac{(A+B+C)x^2 + (-5A-4B-3C)x + (6A+3B+2C)}{(x-1)(x-2)(x-3)}$$

x^2 : $0 = A+B+C$

2. $I = \int \frac{x^4+2}{x^3+2x^2+2x+1}\, dx$

$(x^4+2) : (x^3+2x^2+2x+1) = x-2$

$\quad - - - -$

$\quad\;\; - - - -$

$\qquad 2x^2+3x+4$

$$\Rightarrow \frac{x^4+2}{x^3+2x^2+2x+1} = x-2 + \frac{2x^2+3x+4}{x^3+2x^2+2x+1}$$

$$\Rightarrow I = \int (x-2)\,dx + \int \frac{2x^2+3x+4}{x^3+2x^2+2x+1}$$

$$= \frac{x^2}{2} - 2x + \underbrace{\int \frac{2x^2+3x+4}{(x+1)(x^2+x+1)}}_{J}$$

$$\frac{2x^2+3x+4}{(x+1)(x^2+x+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+x+1}$$

$$= \frac{(A+B)x^2 + (A+B+C)x + (A+C)}{(x+1)(x^2+x+1)}$$

$x^2 : 2 = A+B$

$x^1 : 3 = A+B+C \quad \Rightarrow \quad A = 3, \quad B = -1, \quad C = 1$

$x^0 : 4 = A+C$

$$J = 3\int \frac{dx}{x+1} + \int \frac{-x+1}{x^2+x+1}\,dx$$

$$= 3 \log|x+1| - \frac{1}{2}\int \frac{2x-2}{x^2+x+1}\,dx$$

$$= 3 \log|x+1| - \frac{1}{2}\int \frac{2x+1-3}{x^2+x+1}\,dx$$

$$= 3 \log|x+1| - \frac{1}{2}\int \frac{2x+1}{x^2+x+1} + \frac{3}{2}\int \frac{dx}{x^2+x+1}$$

$$= 3 \log|x+1| - \frac{1}{2}\log(x^2+x+1) + \frac{3}{2}\int \frac{dx}{(x+\frac{1}{2})^2+\frac{3}{4}}$$

$$= 3 \log|x+1| - \frac{1}{2}\log(x^2+x+1) + \sqrt{3}\,\mathrm{arctg}\,\frac{2x+1}{\sqrt{3}} + c$$

3. $\int \frac{x^3+1}{(x^2+1)(x^2+x+1)}dx$

$$\frac{x^3+1}{(x^2+1)(x^2+x+1)} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{x^2+x+1} = \frac{(A+C)x^3+(B+D+A)x^2+(A+B+C)x+B+D}{(x^2+1)(x^2+x+1)}$$

$$\begin{array}{lll} x^3: & 1 = A+C & \\ x^2: & 0 = B+D+A & \\ x: & 0 = A+B+C & \\ x^0: & 1 = B+D & \end{array} \Rightarrow \begin{array}{l} A=-1 \\ B=-1 \\ C=2 \\ D=2 \end{array}$$

$$I = -\int \frac{x+1}{x^2+1}dx + 2\int \frac{x+1}{x^2+x+1}dx$$

$$= -\frac{1}{2}\log(x^2+1) - \operatorname{arctg} x + \log(x^2+x+1) + \frac{2}{\sqrt{3}}\operatorname{arctg}\frac{2x+1}{\sqrt{3}} + c$$

4. $\int \frac{x^2+1}{(x-1)^3}dx$

$$\frac{x^2+1}{(x-1)^3} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3} = \frac{A(x-1)^2+B(x-1)+C}{(x-1)^3}$$

$$= \frac{Ax^2+(-2A+B)x+(A-B+C)}{(x-1)^3}$$

$$\begin{array}{ll} x^2: & 1 = A \\ x: & 0 = -2A+B \\ x^0: & 1 = A-B+C \end{array} \Rightarrow A=1,\ B=2,\ C=2$$

$$I = \int \frac{dx}{x-1} + 2\int\frac{dx}{(x-1)^2} + 2\int\frac{dx}{(x-1)^3} = \ln|x-1| - \frac{2}{x-1} - \frac{1}{(x-1)^2} + C.$$

5. $\int \frac{x^3+1}{(x-1)^2(x^2+1)^2}$

$$\frac{x^3+1}{(x-1)^2(x^2+1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2+1} + \frac{Ex+F}{(x^2+1)^2} \Rightarrow \begin{array}{l} A=\frac{1}{2},\ B=-\frac{1}{4} \\ C=\frac{1}{2},\ D=\frac{1}{2} \\ E=\frac{1}{4},\ F=-\frac{1}{4} \end{array}$$

$$I = \frac{1}{2}\int\frac{dx}{x-1} - \frac{1}{4}\int\frac{dx}{(x-1)^2} + \frac{1}{2}\int\frac{x+1}{x^2+1}dx + \frac{1}{4}\int\frac{x-1}{(x^2+1)^2}dx$$

$$= \frac{1}{2}\ln|x-1| + \frac{1}{4}\frac{1}{x-1} + \frac{1}{4}\int\frac{2x}{x^2+1} + \frac{1}{2}\int\frac{dx}{x^2+1} + \frac{1}{8}\int\frac{2x}{(x^2+1)^2} - \frac{1}{4}\int\frac{dx}{(x^2+1)^2}$$

$$= \frac{1}{2}\ln|x-1| + \frac{1}{4}\frac{1}{x-1} + \frac{1}{4}\ln|x^2+1| + \frac{1}{2}\operatorname{arctg} x - \frac{1}{8}\frac{1}{x^2+1} - \frac{1}{4}\cdot I_2$$

НАПОМЕНА: Погледати рекурентне формуле: $I_n = \int \frac{dx}{\ }$

НАПОМЕНА: **I** Интеграли неких ирационалних функција погодним сменама своди се на рационалну интеграцију.

1. $$\int \frac{x\sqrt{x}}{\sqrt{x}+x}\,dx = \left|\begin{array}{l} \sqrt{x}=t \\ x=t^2 \\ dx=2t\,dt \end{array}\right| = \int \frac{t^2\cdot t}{t+t^2}\cdot 2t\,dt = 2\int \frac{t^3}{1+t}\,dt$$

2. $$\int \frac{\sqrt{x}\,dx}{\sqrt[3]{x}-1} = \left|\begin{array}{ll} \sqrt[6]{x}=t & \sqrt{x}=t^3 \\ x=t^6 & \sqrt[3]{x}=t^2 \\ dx=6t^5dt & \end{array}\right| = \int \frac{t^3\cdot 6t^5dt}{t^2-1} = 6\int \frac{t^8}{t^2-1}$$

II Ако је $R(x)$ рационална функција, тада се интеграл $\int R(e^x)\,dx$ сменом $x=\ln t$ $(t>0)$ своди на интеграл рационалне функције.

1. $$\int \frac{dx}{1+e^{2x}} = \left|\begin{array}{l} e^x=t \\ x=\ln t \\ dx=\frac{dt}{t} \end{array}\right| = \int \frac{dt}{t(1+t^2)}$$

2. $$\int \frac{e^x+1}{e^{2x}+1}\,dx = \left|\begin{array}{l} e^x=t \\ x=\ln t \\ dx=\frac{dt}{t} \end{array}\right| = \int \frac{t+1}{t(t^2+1)}\,dt$$

III Интеграл рационалне ф-је по $\sin x$ и $\cos x$ $\int R(\sin x, \cos x)\,dx$ сменом $t=\operatorname{tg}\frac{x}{2}$ своди се на интеграл рационалне ф-је. При тој смени је

$$\sin x = \frac{2t}{1+t^2}, \quad \cos x = \frac{1-t^2}{1+t^2}, \quad dx = \frac{2dt}{1+t^2}$$

1. $$\int \frac{dx}{1+\sin x-\cos x} = \int \frac{\frac{2dt}{1+t^2}}{1+\frac{2t}{1+t^2}-\frac{1-t^2}{1+t^2}} = \int \frac{\frac{2dt}{1+t^2}}{\frac{1+t^2+2t-1+t^2}{1+t^2}} = \int \frac{dt}{t(t+1)}$$

2. $$\int \frac{dx}{2+\sin x} = \int \frac{\frac{2dt}{1+t^2}}{2+\frac{2t}{1+t^2}} = \int \frac{\frac{2dt}{1+t^2}}{\frac{2+2t^2+2t}{1+t^2}} = \int \frac{dt}{t^2+t+1}$$

3. $$\int \frac{\sin x}{4\sin x - 3\cos x}\,dx = \int \frac{\frac{2t}{1+t^2}\cdot\frac{2dt}{1+t^2}}{4\frac{2t}{1+t^2}-3\frac{1-t^2}{1+t^2}} = \int \frac{2dt}{3t^2+8t-3}$$

РЕКУРЕНТНЕ ФОРМУЛЕ

1. $I_n = \int x^n e^{ax}\,dx \quad , a \neq 0$

$$I_n = \left| \begin{array}{ll} x^n = u & e^{ax}dx = dv \\ du = n x^{n-1} dx & v = \frac{e^{ax}}{a} \end{array} \right|$$

$$= \frac{x^n}{a} e^{ax} - \frac{n}{a} \int x^{n-1} e^{ax}\,dx$$

$$I_n = \frac{x^n}{a} e^{ax} - \frac{n}{a} I_{n-1} \,, \quad I_0 = \frac{e^{ax}}{a} + c$$

2. $I_n = \int \frac{x^n}{\sqrt{x^2+a^2}}\,dx \,, \quad n \geq 2$

$$I_n = \left| \begin{array}{ll} x^{n-1} = u & \frac{x\,dx}{\sqrt{x^2+a}} = dv \\ du = (n-1)\,x^{n-2} dx & v = \sqrt{x^2+a} \end{array} \right|$$

$$= x^{n-1}\sqrt{x^2+a} - (n-1)\int x^{n-2}\sqrt{x^2+a}\,dx$$

$$= x^{n-1}\sqrt{x^2+a} - (n-1)\int \frac{x^{n-2}(x^2+a)}{\sqrt{x^2+a}}\,dx$$

$$= x^{n-1}\sqrt{x^2+a} - (n-1)\int \frac{x^n}{\sqrt{x^2+a}}\,dx - a(n-1)\int \frac{x^{n-2}}{\sqrt{x^2+a}}\,dx$$

$$\Rightarrow \quad I_n = x^{n-1}\sqrt{x^2+a} - (n-1)\,I_n - a(n-1)\,I_{n-2}$$

$$I_n = \frac{x^{n-1}}{n}\sqrt{x^2+a} - a\,\frac{n-1}{n}\,I_{n-2}$$

$$I_1 = \int \frac{x}{\sqrt{x^2+a}}\,dx = \sqrt{x^2+a} + c$$

$$I_0 = \int \frac{dx}{\sqrt{x^2+a}} = \ln\left|x + \sqrt{x^2+a}\right| + c$$

3. $I_n = \int \sin^n x\,dx \quad , n \geq 2$

$$I_n = \left| \begin{array}{ll} \sin^{n-1} x = u & \sin x\,dx = dv \\ du = (n-1)\sin^{n-2}x \cos x\,dx & v = -\cos x \end{array} \right|$$

$$= -\cos x \sin^{n-1} x + (n-1)\int \sin^{n-2} x \cos^2 x\,dx$$

$$= -\cos x \sin^{n-1} x + (n-1)\int \sin^{n-2} x\,(1-\sin^2 x)\,dx$$

$$= -\cos x \sin^{n-1} x + (n-1)\,I_{n-2} - (n-1)\,I_n$$

4. $I_n = \int \cos^n x \, dx = \left| \begin{array}{ll} \cos^{n-1} x = u & \cos x \, dx = dv \\ du = -(n-1)\cos^{n-2} x \sin x \, dx & v = \sin x \end{array} \right|$

$$= \sin x \cos^{n-1} x + (n-1) \int \cos^{n-2} x \sin^2 x \, dx$$

$$= \sin x \cos^{n-1} x + (n-1) \int \cos^{n-2} x \, (1 - \cos^2 x) \, dx$$

$$= \sin x \cos^{n-1} x + (n-1) I_{n-2} - (n-1) I_n$$

$\Rightarrow I_n = \frac{n-1}{n} I_{n-2} + \sin x \cos^{n-1} x \quad , \quad I_1 = \sin x + c \quad , I_0 = x + c$

напомена: 4. се може добити из 3. увођењем смене $x = y - \frac{\pi}{2}$

5. $I_n = \int \frac{dx}{\sin^n x} \quad , n \geq 2$

$I_n = \int \frac{\sin x}{\sin^{n+1} x} \, dx = \left| \begin{array}{ll} \sin^{-(n+1)} x = u & \sin x \, dx = dv \\ du = -(n+1) \sin^{-(n+2)} \cos x \, dx & v = -\cos x \end{array} \right|$

$$= - \frac{\cos x}{\sin^{n+1} x} - (n+1) \int \frac{\cos^2 x}{\sin^{n+2} x} \, dx$$

$$= - \frac{\cos x}{\sin^{n+1} x} - (n+1) \int \frac{1 - \sin^2 x}{\sin^{n+2} x} \, dx$$

$$= - \frac{\cos x}{\sin^{n+1} x} - (n+1) I_{n+2} + (n+1) I_n$$

$\Rightarrow (n+1) I_{n+2} = n I_n - \frac{\cos x}{\sin^{n+1} x}$

$\Rightarrow (n-1) I_n = (n-2) I_{n-2} - \frac{\cos x}{\sin^{n-1} x}$

$I_1 = \log \left| \operatorname{tg} \frac{x}{2} \right| + c \quad , \quad I_0 = x + c$

6. $I_n = \int \operatorname{tg}^n x \, dx \quad , n \geq 2$

$I_n = \int \operatorname{tg}^{n-2} x \cdot \operatorname{tg}^2 x \, dx = \int \operatorname{tg}^{n-2} x \cdot \frac{1 - \cos^2 x}{\cos^2 x} \, dx = \int \frac{\operatorname{tg}^{n-2} x}{\cos^2 x} \, dx - I_{n-2}$

$\Rightarrow I_n = \frac{\operatorname{tg}^{n-1} x}{n-1} - I_{n-2} \quad , \quad I_1 = -\log|\cos x| + c \quad , \quad I_0 = x + c$

7. $I_n = \int \frac{dx}{(x^2+a^2)^n} \quad , \; n \geq 2$

$$I_n = \frac{1}{a^2} \int \frac{a^2\,dx}{(x^2+a^2)^n} = \frac{1}{a^2} \int \frac{x^2+a^2-x^2}{(x^2+a^2)^n}\,dx$$

$$= \frac{1}{a^2} \int \frac{dx}{(x^2+a^2)^{n-1}} - \frac{1}{a^2} \int \frac{x^2}{(x^2+a^2)^n}\,dx$$

$$= \frac{1}{a^2} I_{n-1} - \frac{1}{a^2} \int \frac{x^2}{(x^2+a^2)^n}\,dx = \left| \begin{array}{ll} x = u & \frac{x}{(x^2+a^2)^n}dx = dv \\ du = dx & v = -\frac{(x^2+a^2)^{-n+1}}{2(n-1)} \end{array} \right|$$

$$= -\frac{1}{a^2} I_{n-1} - \frac{1}{a^2}\left(-\frac{x}{2(n-1)(x^2+a^2)^{n-1}} + \frac{1}{2(n-1)} \int \frac{dx}{(x^2+a^2)^{n-1}} \right)$$

$$= \frac{1}{a^2} I_{n-1} + \frac{1}{a^2} \cdot \frac{x}{2(n-1)(x^2+a^2)^{n-1}} - \frac{1}{2a^2(n-1)} I_{n-1}$$

$$\Rightarrow I_n = \frac{2n-3}{2n-2} \cdot \frac{1}{a^2} I_{n-1} + \frac{1}{2a^2(n-1)} \cdot \frac{x}{(x^2+a^2)^{n-1}}$$

$$I_1 = \frac{1}{a} \operatorname{arctg} \frac{x}{a} + c \quad , \quad I_0 = x + c$$

8. $I_n = \int \frac{dx}{(ax^2+bx+c)^n} \quad , \; a \neq 0$

$$ax^2+bx+c = a(x-\alpha)^2 + \beta \, , \quad \alpha = -\frac{b}{2a} \; , \; \beta = \frac{4ac-b^2}{4a}$$

$$= \frac{1}{4a}\left[(2ax+b)^2 + 4ac - b^2 \right] = \frac{1}{4a}(t^2+\Delta)$$

$2ax + b = t \quad , \quad \Delta = 4ac - b^2$

$2a\,dx = dt$

$$I_n = \frac{(4a)^n}{2a} \int \frac{dt}{(t^2+\Delta)^n} = \frac{(4a)^n}{2a} J_n$$

$$\overset{7.}{=} \frac{(4a)^n}{2a} \cdot \left(\frac{2n-3}{2n-2} \cdot \frac{1}{\Delta} J_{n-1} + \frac{1}{2\Delta(n-1)} \cdot \frac{t}{(t^2+\Delta)^{n-1}} \right)$$

$$I_{n-1} = \frac{(4a)^{n-1}}{2a} J_{n-1}$$

$$= \frac{4a}{\Delta} \cdot \frac{2n-3}{2n-2} \cdot I_{n-1} + \frac{2ax+b}{(ax^2+bx+c)^{n-1}} \cdot \frac{1}{\delta(n-1)}$$

ИНТЕГРАЦИЈА ИРАЦИОНАЛНИХ Ф-ЈА

I Метод Остроградског

$$\int \frac{P_n(x)}{\sqrt{ax^2+bx+c}}\,dx = Q_{n-1}(x)\sqrt{ax^2+bx+c} + \lambda \int \frac{dx}{\sqrt{ax^2+bx+c}}$$

где је $P_n(x)$ полином степена n и
$Q_{n-1}(x)$ полином степена $n-1$ са неодређеним коеф. и λ је константа.

Коефицијенти полинома $Q_{n-1}(x)$ и λ одређују се диференцирањем дате једнакости.

1. $$\int \frac{x^2+x+2}{\sqrt{x^2+x+1}}\,dx = (Ax+B)\sqrt{x^2+x+1} + \lambda\int\frac{dx}{\sqrt{x^2+x+1}} \quad /'$$

$$\frac{x^2+x+2}{\sqrt{x^2+x+1}} = A\sqrt{x^2+x+1} + (Ax+B)\frac{2x+1}{2\sqrt{x^2+x+1}} + \frac{\lambda}{\sqrt{x^2+x+1}} \quad / \cdot 2\sqrt{x^2+x+1}$$

$$2(x^2+x+2) = 2A(x^2+x+1) + (Ax+B)(2x+1) + 2\lambda$$

$$2x^2+2x+4 = 4Ax^2 + (3A+2B)x + (2A+B+2\lambda)$$

$$\Rightarrow \quad \begin{aligned} x^2 &: \ 2 = 4A \\ x &: \ 2 = 3A+2B \\ x^0 &: \ 4 = 2A+B+2\lambda \end{aligned} \quad \Rightarrow \quad A = \frac{1}{2}, \ B = \frac{1}{4}, \ \lambda = \frac{11}{8}$$

Дакле,

$$\int \frac{x^2+x+2}{\sqrt{x^2+x+2}}\,dx = \left(\frac{1}{2}x+\frac{1}{4}\right)\sqrt{x^2+x+1} + \frac{11}{8}\int\frac{dx}{\sqrt{x^2+x+1}}$$

$$= \left(\frac{1}{2}x+\frac{1}{4}\right)\sqrt{x^2+x+1} + \frac{11}{8}\int\frac{dx}{\sqrt{(x+\frac{1}{2})^2+\frac{3}{4}}}$$

$$= \left(\frac{1}{2}x+\frac{1}{4}\right)\sqrt{x^2+x+1} + \frac{11}{8}\ln\left|2x+1+\sqrt{x^2+x+1}\right| + C$$

2. $$\int\sqrt{x^2+1}\,dx = \int\frac{x^2+1}{\sqrt{x^2+1}}\,dx = (Ax+B)\sqrt{x^2+1} + \lambda\int\frac{dx}{\sqrt{x^2+1}}$$

$A = 1/2$, $B = 0$, $\lambda = 1/2$

II Интеграл облика $\int \frac{Ax+B}{(x-\alpha)^n\sqrt{ax^2+bx+c}}\,dx$

израчунава се сменом $x-\alpha=\frac{1}{t}$.

4. $\int \frac{dx}{(x+1)^5\sqrt{x^2+2x}} = \left| \begin{array}{l} x+1=\frac{1}{t} \\ x=\frac{1-t}{t} \end{array} \right| = \int \frac{t^4\,dt}{\sqrt{1-t^2}} \Rightarrow$ случај I

III Биномни диференцијал:

$$I=\int x^m(a+bx^n)^p\,dx \quad , \ m,n,p\in\mathbb{Q}, \ a,b\in\mathbb{R}$$

<u>Први случај:</u> p – цео број

- за $p>0$ бином $(a+bx^n)^p$ треба развити по биномном обрасцу

5. $\int x^{3/2}(1-x^{1/2})^2\,dx = \int x^{3/2}\cdot(1-2x^{1/2}+x)\,dx = \int(x^{\frac{3}{2}}+2x^2+x^{5/2})\,dx$

- за $p<0$ смена $x=t^s$, где је s НЗС за имениоце разломака m и n

6. $\int x^{-1/4}\,(1+x^{1/6})^{-2}\,dx$

$m=-\frac{1}{4}$, $n=\frac{1}{6}$, $p=-2$ цео негативан број

смена: $x=t^{12} \Rightarrow I=\int t^{-3}\,(1-t^2)^{-2}\cdot 12t^{11}\,dt$ рац. инт.

<u>Други случај:</u> $\frac{m+1}{n}$ је цео број

смена $a+bx^n=t^r$, где је r именилац разломка p

7. $I=\int \frac{\sqrt[3]{1+\sqrt[4]{x}}}{\sqrt{x}}\,dx = \int x^{-\frac{1}{2}}\,(1+x^{1/4})^{1/3}\,dx$

$m=-\frac{1}{2}$, $n=\frac{1}{4}$, $p=\frac{1}{3}$, $\frac{m+1}{n}=2$ – цео број

смена: $1+x^{1/4}=t^3$, $dx=12t^2(t^3-1)^3\,dt$

$I=12\int(t^6-t^3)\,dt$

Трећи случај: $\frac{m+1}{n}+p$ је цео број

смена: $ax^{-n}+b=t^{s}$, где је s именилац разломка p.

8. $I=\int x^4(1-x^2)^{-\frac{3}{2}}dx$

$m=4,\ n=2,\ p=-\frac{3}{2}$, $\frac{m+1}{n}+p=1$ цео број

$$I=\left|x^{-2}-1=t^2\right| = -\int\frac{dt}{t^2(t^2+1)^2}$$

9. $I=\int\frac{dx}{\sqrt[4]{1+x^4}}=\int x^0(1+x^4)^{-1/4}dx$

$\frac{m+1}{n}+p=0$ цео број $\Rightarrow$ смена $1+x^{-4}=t^4$

$x=(t^4-1)^{-\frac{1}{4}}$

$dx=-t^3(t^4-1)^{-5/4}dt$

$$I=-\int\frac{t^2dt}{t^4-1}$$

IV Ојлерове смене

$$I=\int R\left(x,\sqrt{ax^2+bx+c}\right)dx$$

Први случај: Ако је $a>0$ смена $\sqrt{ax^2+bx+c}=t-x\sqrt{a}$

или $t+x\sqrt{a}$

10. $$I=\int\frac{dx}{x\sqrt{4x^2+4x+3}}=\left|\begin{array}{l}\sqrt{4x^2+4x+3}=t-2x\\ x=\frac{t^2-3}{4(1+t)},\quad dx=\frac{t^2+2t+3}{4(1+t)^2}dt\end{array}\right|$$

$$=2\int\frac{dt}{t^2-3}=\frac{1}{\sqrt{3}}\ln\left|\frac{t-\sqrt{3}}{t+\sqrt{3}}\right|+c$$

$$=\frac{1}{\sqrt{3}}\ln\left|\frac{2x-\sqrt{3}+\sqrt{4x^2+4x+3}}{2x+\sqrt{3}+\sqrt{4x^2+4x+3}}\right|+c$$

Други случај: Ако је $c>0$, смена $\sqrt{ax^2+bx+c} = tx - \sqrt{c}$

или $tx + \sqrt{c}$

11. $$\int \frac{dx}{(1+x)\sqrt{1+x-x^2}} = \left| \begin{array}{l} \sqrt{1+x-x^2} = tx - 1 \\ \cancel{1}+x-x^2 = t^2x^2 - 2tx + \cancel{1} \\ x(1-x) = x(t^2x - 2t) \\ 1-x = t^2x - 2t \\ x(t^2+1) = 1+2t \\ x = \frac{1+2t}{t^2+1} \end{array} \right. \quad dx = -\frac{2(t^2+t-1)}{(t^2+1)^2}dt$$

$$I = -2\int \frac{dt}{1+(t+1)^2} = -2\,\mathrm{arctg}\,(t+1) + c$$

$$= -2\,\mathrm{arctg}\,\frac{1+x+\sqrt{1+x-x^2}}{x} + c$$

Трећи случај: Ако су корени квадратног тринома ax^2+bx+c реални

$$\sqrt{ax^2+bx+c} = \sqrt{a(x-\alpha)(x-\beta)} = (x-\alpha)t$$

12. $$I = \int \frac{(x-1)\,dx}{(x^2+2x)\sqrt{x^2+2x}}$$

Квадратни трином x^2+2x има корене $\alpha=0$ и $\beta=-2$ па се може увести смена $\sqrt{x^2+2x} = xt$

$$x = \frac{2}{t^2-1}, \quad dx = -\frac{4t\,dt}{(t^2-1)^2}$$

$$I = -\frac{1}{2}\int \frac{3-t^2}{t^2}dt = \frac{3}{2t} - \frac{1}{2}t + c = \frac{1+2x}{\sqrt{x^2+2x}} + c$$

НАПОМЕНА: Случајеви $a>0$ и $c>0$ своде се један на други сменом $x=\frac{1}{t}$.

V У следећим интегралима може се користити неке од доле наведених смена.

за $\sqrt{a^2-b^2x^2}$ смена $x=\frac{a}{b}\sin t$ доводи до $a\sqrt{1-\sin^2 t} = a\cdot\cos t$

за $\sqrt{a^2+b^2x^2}$ смена $x=\frac{a}{b}\,\mathrm{tg}\,t$ доводи до $a\sqrt{1+\mathrm{tg}^2 t} = \frac{a}{\cos t}$