Granične vrednosti funkcija

Granične vrednosti funkcija

Poznate granične vrednosti:

1. $\lim\limits_{x\to 0} \dfrac{\sin x}{x} = 1;$

2. $\lim\limits_{x\to 0} (1+x)^{\frac{1}{x}} = e \quad \Leftrightarrow \quad \lim\limits_{x\to\infty} \Big(1+\dfrac{1}{x}\Big)^{x} = e.$

3. $\lim\limits_{x\to 0} \dfrac{e^x - 1}{x} = 1;$

4. $\lim\limits_{x\to 0} \dfrac{\log(1+x)}{x} = 1, \qquad (\log = \log_e);$

Primer 1. a) $\lim\limits_{x\to 0} \dfrac{\sin ax}{\sin bx} = \lim\limits_{x\to 0} \dfrac{\frac{\sin ax}{ax}\, ax}{\frac{\sin bx}{bx}\, bx} = \dfrac{a}{b};$

b) $\lim\limits_{x\to 0} \dfrac{\sin^2 ax}{x^2} = \lim\limits_{x\to 0} \Big(\dfrac{\sin ax}{ax}\Big)^2 a^2 = a^2;$

c) $\lim\limits_{x\to 0} \dfrac{1-\cos kx}{x^2} = \lim\limits_{x\to 0} \dfrac{2\sin^2 \frac{kx}{2}}{x^2} = \lim\limits_{x\to 0} \Big(\dfrac{\sin \frac{kx}{2}}{\frac{kx}{2}}\Big)^2 \dfrac{k^2}{2} = \dfrac{k^2}{2};$

Zadaci:

1. Odrediti:

a) $\lim\limits_{x\to+\infty}\sqrt{x}\Big(\sqrt{x+2}-\sqrt{x+1}\Big)$;

b) $\lim\limits_{x\to+\infty}\Big(\sqrt{2x+\sqrt{x+\sqrt{x}}}-\sqrt{2x}\Big)$;

c) $\lim\limits_{x\to1}\dfrac{\sqrt[3]{x^2}-2\sqrt[3]{x}+1}{(x-1)^2}$;

d) $\lim\limits_{x\to+\infty}\dfrac{\sqrt[3]{1+\dfrac{4}{x}}-\sqrt[3]{1+\dfrac{3}{x}}}{1-\sqrt{1+\dfrac{5}{x}}}$;

e) $\lim\limits_{x\to1}\Big(\dfrac{x+2}{x^2-5x+4}-\dfrac{1}{x^2-3x+2}\Big)$;

f) $\lim\limits_{x\to2}\dfrac{\sqrt{1+4x}-3}{\sqrt{2x}-2}$;

g) $\lim\limits_{x\to1}\dfrac{\sqrt[3]{8x}-2}{\sqrt{x^2+3}-2}$;

Rešenje: a) Imamo

$$\begin{aligned}
L &= \lim_{x\to+\infty}\sqrt{x}\Big(\sqrt{x+2}-\sqrt{x+1}\Big)\\
&= \lim_{x\to+\infty}\sqrt{x}\Big(\sqrt{x+2}-\sqrt{x+1}\Big)\frac{\sqrt{x+2}+\sqrt{x+1}}{\sqrt{x+2}+\sqrt{x+1}} = \lim_{x\to+\infty}\sqrt{x}\frac{x+2-(x+1)}{\sqrt{x+2}+\sqrt{x+1}}\\
&= \lim_{x\to+\infty}\frac{\sqrt{x}}{\sqrt{x}\Big(\sqrt{1+\frac{2}{x}}+\sqrt{1+\frac{1}{x}}\Big)} = \lim_{x\to+\infty}\frac{1}{\sqrt{1+\frac{2}{x}}+\sqrt{1+\frac{1}{x}}} = \frac{1}{2}.
\end{aligned}$$

b) Računamo

$$\begin{aligned}
L &= \lim_{x\to+\infty}\Big(\sqrt{2x+\sqrt{x+\sqrt{x}}}-\sqrt{2x}\Big)\\
&= \lim_{x\to+\infty}\Big(\sqrt{2x+\sqrt{x+\sqrt{x}}}-\sqrt{2x}\Big)\frac{\sqrt{2x+\sqrt{x+\sqrt{x}}}+\sqrt{2x}}{\sqrt{2x+\sqrt{x+\sqrt{x}}}+\sqrt{2x}}\\
&= \lim_{x\to+\infty}\frac{2x+\sqrt{x+\sqrt{x}}-2x}{\sqrt{2x+\sqrt{x+\sqrt{x}}}+\sqrt{2x}} = \lim_{x\to+\infty}\frac{\sqrt{x+\sqrt{x}}}{\sqrt{2x+\sqrt{x+\sqrt{x}}}+\sqrt{2x}}\\
&= \lim_{x\to+\infty}\frac{\sqrt{x}\sqrt{1+\frac{1}{\sqrt{x}}}}{\sqrt{x}\Big(\sqrt{2+\sqrt{\frac{1}{x}+\frac{1}{x\sqrt{x}}}}+\sqrt{2}\Big)} = \lim_{x\to+\infty}\frac{\sqrt{1+\frac{1}{\sqrt{x}}}}{\Big(\sqrt{2+\sqrt{\frac{1}{x}+\frac{1}{x\sqrt{x}}}}+\sqrt{2}\Big)} = \frac{1}{2\sqrt{2}}.
\end{aligned}$$

c) Nalazimo

$$L = \lim_{x\to 1} \frac{\sqrt[3]{x^2} - 2\sqrt[3]{x} + 1}{(x-1)^2} = \lim_{x\to 1} \frac{\left(\sqrt[3]{x} - 1\right)^2}{(x-1)^2} = \lim_{x\to 1} \frac{\left(\sqrt[3]{x} - 1\right)^2}{\Big(\left(\sqrt[3]{x} - 1\right)\left(\sqrt[3]{x^2} + \sqrt[3]{x} + 1\right)\Big)^2}$$

$$= \lim_{x\to 1} \frac{1}{\left(\sqrt[3]{x^2} + \sqrt[3]{x} + 1\right)^2} = \frac{1}{9}.$$

d) Određujemo

$$L = \lim_{x\to +\infty} \frac{\sqrt[3]{1+\frac{4}{x}} - \sqrt[3]{1+\frac{3}{x}}}{1 - \sqrt{1+\frac{5}{x}}} = \left| \begin{array}{c} a^3 - b^3 = (a-b)(a^2+ab+b^2) \\ a^2 - b^2 = (a-b)(a+b) \end{array} \right|$$

$$= \lim_{x\to +\infty} \frac{\sqrt[3]{1+\frac{4}{x}} - \sqrt[3]{1+\frac{3}{x}}}{1 - \sqrt{1+\frac{5}{x}}} \cdot \frac{\sqrt[3]{\left(1+\frac{4}{x}\right)^2} + \sqrt[3]{\left(1+\frac{4}{x}\right)\left(1+\frac{3}{x}\right)} + \sqrt[3]{\left(1+\frac{3}{x}\right)^2}}{\sqrt[3]{\left(1+\frac{4}{x}\right)^2} + \sqrt[3]{\left(1+\frac{4}{x}\right)\left(1+\frac{3}{x}\right)} + \sqrt[3]{\left(1+\frac{3}{x}\right)^2}}$$

$$\times \frac{1 + \sqrt{1+\frac{5}{x}}}{1 + \sqrt{1+\frac{5}{x}}}$$

$$= \lim_{x\to +\infty} \frac{1 + \frac{4}{x} - \left(1+\frac{3}{x}\right)}{1 - \left(1+\frac{5}{x}\right)} \cdot \frac{1 + \sqrt{1+\frac{5}{x}}}{\sqrt[3]{\left(1+\frac{4}{x}\right)^2} + \sqrt[3]{\left(1+\frac{4}{x}\right)\left(1+\frac{3}{x}\right)} + \sqrt[3]{\left(1+\frac{3}{x}\right)^2}}$$

$$= \lim_{x\to +\infty} \frac{\frac{1}{x}}{-\frac{5}{x}} \cdot \frac{1 + \sqrt{1+\frac{5}{x}}}{\sqrt[3]{\left(1+\frac{4}{x}\right)^2} + \sqrt[3]{\left(1+\frac{4}{x}\right)\left(1+\frac{3}{x}\right)} + \sqrt[3]{\left(1+\frac{3}{x}\right)^2}} = -\frac{2}{15}.$$

e) Imamo

$$L = \lim_{x\to 1} \left(\frac{x+2}{x^2 - 5x + 4} - \frac{1}{x^2 - 3x + 2} \right) = \lim_{x\to 1} \left(\frac{x+2}{(x-1)(x-4)} - \frac{1}{(x-1)(x-2)} \right)$$

$$= \lim_{x\to 1} \frac{x^2 - 4 - (x-4)}{(x-1)(x-2)(x-4)} = \lim_{x\to 1} \frac{x(x-1)}{(x-1)(x-2)(x-4)}$$

$$= \lim_{x\to 1} \frac{x}{(x-2)(x-4)} = \frac{1}{3}.$$

f) Nalazimo

$$\begin{aligned}
L &= \lim_{x\to 2}\frac{\sqrt{1+4x}-3}{\sqrt{2x}-2} = \lim_{x\to 2}\frac{\sqrt{1+4x}-3}{\sqrt{2x}-2}\cdot\frac{\sqrt{1+4x}+3}{\sqrt{1+4x}+3}\cdot\frac{\sqrt{2x}+2}{\sqrt{2x}+2}\\
&= \lim_{x\to 2}\frac{1+4x-9}{2x-4}\cdot\frac{\sqrt{2x}+2}{\sqrt{1+4x}+3} = \lim_{x\to 2}\frac{4(x-2)}{2(x-2)}\cdot\frac{\sqrt{2x}+2}{\sqrt{1+4x}+3}\\
&= \lim_{x\to 2} 2\frac{\sqrt{2x}+2}{\sqrt{1+4x}+3} = \frac{4}{3}.
\end{aligned}$$

g) Određujemo

$$\begin{aligned}
L &= \lim_{x\to 1}\frac{\sqrt[3]{8x}-2}{\sqrt{x^2+3}-2} = \lim_{x\to 1}\frac{\sqrt[3]{8x}-2}{\sqrt{x^2+3}-2}\cdot\frac{\sqrt[3]{(8x)^2}+2\sqrt[3]{8x}+4}{\sqrt[3]{(8x)^2}+2\sqrt[3]{8x}+4}\cdot\frac{\sqrt{x^2+3}+2}{\sqrt{x^2+3}+2}\\
&= \lim_{x\to 1}\frac{8x-8}{x^2+3-4}\cdot\frac{\sqrt{x^2+3}+2}{\sqrt[3]{(8x)^2}+2\sqrt[3]{8x}+4}\\
&= \lim_{x\to 1}\frac{8(x-1)}{(x-1)(x+1)}\cdot\frac{\sqrt{x^2+3}+2}{\sqrt[3]{(8x)^2}+2\sqrt[3]{8x}+4} = \lim_{x\to 1}\frac{8}{x+1}\cdot\frac{\sqrt{x^2+3}+2}{\sqrt[3]{(8x)^2}+2\sqrt[3]{8x}+4}\\
&= \frac{4}{3}.
\end{aligned}$$

2. Neka je

$$f(x) = \frac{27x^3-4x^2+2013\sin x}{2013x^3-4x^2+27x}.$$

Odrediti

$$\lim_{x\to+\infty} f(x), \qquad \lim_{x\to 0} f(x);$$

Rešenje: Računamo

$$\begin{aligned}
L_1 &= \lim_{x\to+\infty}\frac{27x^3-4x^2+2013\sin x}{2013x^3-4x^2+27x} = \lim_{x\to+\infty}\frac{x^3\left(27-4\frac{1}{x}+2013\frac{\sin x}{x^3}\right)}{x^3\left(2013-4\frac{1}{x}+27\frac{1}{x^2}\right)}\\
&= \lim_{x\to+\infty}\frac{27-4\frac{1}{x}+2013\frac{\sin x}{x^3}}{2013-4\frac{1}{x}+27\frac{1}{x^2}} = \frac{27}{2013}.
\end{aligned}$$

$$
\begin{aligned}
L_2 &= \lim_{x\to 0} \frac{27x^3 - 4x^2 + 2013\sin x}{2013x^3 - 4x^2 + 27x} = \lim_{x\to 0} \frac{x(27x^2 - 4x + 2013\frac{\sin x}{x})}{x(2013x^2 - 4x + 27)} \\
&= \lim_{x\to 0} \frac{27x^2 - 4x + 2013\frac{\sin x}{x}}{2013x^2 - 4x + 27} = \frac{2013}{27}.
\end{aligned}
$$

3. Odrediti graničnu vrednost

$$\lim_{x\to 1} \frac{\sqrt[k]{x} - 1}{x - 1},$$

a zatim naći

$$\lim_{x\to 1} \frac{(\sqrt{x} - 1)(\sqrt[3]{x} - 1)\ldots(\sqrt[10]{x} - 1)}{(x-1)^9};$$

Rešenje: Važi:

$$a^k - b^k = (a-b)(a^{k-1} + a^{k-2}b + \ldots + ab^{k-2} + b^{k-1}).$$

Imamo

$$
\begin{aligned}
L &= \lim_{x\to 1} \frac{\sqrt[k]{x} - 1}{x - 1} = \lim_{x\to 1} \frac{\sqrt[k]{x} - 1}{(\sqrt[k]{x} - 1)(\sqrt[k]{x^{k-1}} + \sqrt[k]{x^{k-2}} + \ldots + \sqrt[k]{x} + 1)} \\
&= \lim_{x\to 1} \frac{1}{\sqrt[k]{x^{k-1}} + \sqrt[k]{x^{k-2}} + \ldots + \sqrt[k]{x} + 1} = \frac{1}{k}.
\end{aligned}
$$

Sada je

$$
\begin{aligned}
L_1 &= \lim_{x\to 1} \frac{(\sqrt{x} - 1)(\sqrt[3]{x} - 1)\ldots(\sqrt[10]{x} - 1)}{(x-1)^9} = \lim_{x\to 1} \frac{\sqrt{x} - 1}{x - 1} \frac{\sqrt[3]{x} - 1}{x - 1} \ldots \frac{\sqrt[10]{x} - 1}{x - 1} \\
&= \frac{1}{2 \cdot 3 \cdot \ldots \cdot 10} = \frac{1}{10!}.
\end{aligned}
$$

4. Neka je

$$f(x) = \frac{\sqrt{x^2 + 14} + x}{\sqrt{x^2 - 2} + x}.$$

Odrediti granične vrednosti

$$\lim_{x\to +\infty} f(x) \quad \text{i} \quad \lim_{x\to -\infty} f(x);$$

Rešenje: Neka je $L_1 = \lim\limits_{x\to+\infty} f(x)$ i $L_2 = \lim\limits_{x\to-\infty} f(x)$.
Imamo

$$L_1 = \lim_{x\to+\infty} \frac{\sqrt{x^2+14}+x}{\sqrt{x^2-2}+x} = \lim_{x\to+\infty} \frac{x\Big(\sqrt{1+\frac{14}{x^2}}+1\Big)}{x\Big(\sqrt{1-\frac{2}{x^2}}+1\Big)} = \lim_{x\to+\infty} \frac{\sqrt{1+\frac{14}{x^2}}+1}{\sqrt{1-\frac{2}{x^2}}+1} = 1.$$

$$\begin{aligned}
L_2 &= \lim_{x\to-\infty} \frac{\sqrt{x^2+14}+x}{\sqrt{x^2-2}+x} = \lim_{x\to-\infty} \frac{\sqrt{x^2+14}+x}{\sqrt{x^2-2}+x} \cdot \frac{\sqrt{x^2+14}-x}{\sqrt{x^2+14}-x} \cdot \frac{\sqrt{x^2-2}-x}{\sqrt{x^2-2}-x} \\
&= \lim_{x\to-\infty} \frac{x^2+14-x^2}{x^2-2-x^2} \cdot \frac{\sqrt{x^2-2}-x}{\sqrt{x^2+14}-x} = \lim_{x\to-\infty} \frac{14}{-2} \cdot \frac{\sqrt{x^2-2}-x}{\sqrt{x^2+14}-x} \\
&= = \lim_{x\to-\infty} \frac{14}{-2} \cdot \frac{|x|\sqrt{1-\frac{2}{x^2}}-x}{|x|\sqrt{1-\frac{14}{x^2}}-x} = \Big| \, |x| = -x, \quad x \to -\infty \Big| \\
&= \lim_{x\to-\infty} (-7) \frac{-x(\sqrt{1+\frac{14}{x^2}}+1)}{-x(\sqrt{1-\frac{2}{x^2}}+1)} = \lim_{x\to-\infty} (-7) \frac{\sqrt{1+\frac{14}{x^2}}+1}{\sqrt{1-\frac{2}{x^2}}+1} = -7.
\end{aligned}$$

5. Neka je

$$f(x) = \frac{\sqrt{x^2-2x+6}-\sqrt{x^2+2x-6}}{x^2-4x+3}.$$

Odrediti granične vrednosti

$$\lim_{x\to 3} f(x), \qquad \lim_{x\to+\infty} f(x), \qquad \lim_{x\to-\infty} f(x);$$

Rešenje: Određujemo

$$\begin{aligned}
&\frac{\sqrt{x^2-2x+6}-\sqrt{x^2+2x-6}}{x^2-4x+3} \\
&= \frac{\sqrt{x^2-2x+6}-\sqrt{x^2+2x-6}}{x^2-4x+3} \cdot \frac{\sqrt{x^2-2x+6}+\sqrt{x^2+2x-6}}{\sqrt{x^2-2x+6}+\sqrt{x^2+2x-6}} \\
&= \frac{x^2-2x+6-(x^2+2x-6)}{x^2-4x+3} \cdot \frac{1}{\sqrt{x^2-2x+6}+\sqrt{x^2+2x-6}} \\
&= \frac{-4(x-3)}{(x-3)(x-1)} \cdot \frac{1}{\sqrt{x^2-2x+6}+\sqrt{x^2+2x-6}} \\
&= \frac{-4}{(x-1)} \cdot \frac{1}{\sqrt{x^2-2x+6}+\sqrt{x^2+2x-6}} = -\frac{1}{3},
\end{aligned}$$

$$
\begin{aligned}
L_1 &= \lim_{x\to 3}\frac{\sqrt{x^2-2x+6}-\sqrt{x^2+2x-6}}{x^2-4x+3}\\
&= \lim_{x\to 3}\frac{-4}{(x-1)}\cdot\frac{1}{\sqrt{x^2-2x+6}+\sqrt{x^2+2x-6}}=-\frac{1}{3},
\end{aligned}
$$

$$
\begin{aligned}
L_2 &= \lim_{x\to +\infty}\frac{\sqrt{x^2-2x+6}-\sqrt{x^2+2x-6}}{x^2-4x+3}\\
&= \lim_{x\to +\infty}\frac{-4}{(x-1)}\cdot\frac{1}{\sqrt{x^2-2x+6}+\sqrt{x^2+2x-6}}=0,
\end{aligned}
$$

$$
\begin{aligned}
L_3 &= \lim_{x\to -\infty}\frac{\sqrt{x^2-2x+6}-\sqrt{x^2+2x-6}}{x^2-4x+3}\\
&= \lim_{x\to -\infty}\frac{-4}{(x-1)}\cdot\frac{1}{\sqrt{x^2-2x+6}+\sqrt{x^2+2x-6}}=0.
\end{aligned}
$$

6. Odrediti

a) $\lim\limits_{x\to 0}(1-\cos x)\cot x$;

b) $\lim\limits_{x\to 0}\dfrac{\cos 2x-\cos 3x}{\sin^2 5x}$;

c) $\lim\limits_{x\to 0}\dfrac{\tan x-\sin x}{(1+\sin x)\sin^3 x}$;

d) $\lim\limits_{x\to 0}\dfrac{1-\cos^3 x}{x\sin 2x}$;

e) $\lim\limits_{x\to 0}\dfrac{1-\cos(1-\cos 2x)}{x^4}$;

f) $\lim\limits_{x\to 0+}\dfrac{\sqrt{1-\cos x}}{1-\cos\sqrt{x}}$;

g) $\lim\limits_{x\to 0}\dfrac{x}{\sqrt{1+2x+\sin x}-\sqrt{\cos x}}$;

h) $\lim\limits_{x\to \pi/4}\tan 2x\tan\Big(\dfrac{\pi}{4}-x\Big)$;

i) $\lim\limits_{x\to \pi/2}\dfrac{1-\sin x}{\left(\frac{\pi}{2}-x\right)^2}$;

j) $\lim\limits_{x\to 0}\dfrac{\sqrt{2}-\sqrt{1+\cos x}}{\sin^2 x}$;

Rešenje: a) Određujemo

$$
L=\lim_{x\to 0}(1-\cos x)\cot x=\lim_{x\to 0}\frac{(1-\cos x)\cos x}{\sin x}=\lim_{x\to 0}\frac{2\sin^2\dfrac{x}{2}\cos x}{2\sin\dfrac{x}{2}\cos\dfrac{x}{2}}
$$

$$
=\lim_{x\to 0}\frac{\sin\dfrac{x}{2}\cos x}{\cos\dfrac{x}{2}}=0.
$$

b) *I način:* Određujemo

$$
\begin{aligned}
L &= \lim_{x\to 0}\frac{\cos 2x-\cos 3x}{\sin^2 5x}=\lim_{x\to 0}\frac{2\sin\frac{5x}{2}\sin\frac{x}{2}}{\sin^2 5x}\\
&=\lim_{x\to 0}\frac{2\frac{\sin\frac{5x}{2}}{\frac{5x}{2}}\cdot\frac{5x}{2}\cdot\frac{\sin\frac{x}{2}}{\frac{x}{2}}\cdot\frac{x}{2}}{\left(\frac{\sin 5x}{5x}\right)^2 25x^2}=\frac{1}{10}.
\end{aligned}
$$

II način: Određujemo

$$
\begin{aligned}
L &= \lim_{x\to 0}\frac{\cos 2x-\cos 3x}{\sin^2 5x}=\lim_{x\to 0}\frac{(\cos 2x-1)+(1-\cos 3x)}{\sin^2 5x}\\
&=\lim_{x\to 0}\frac{-2\sin^2 x+2\sin^2\frac{3x}{2}}{\sin^2 5x}=\lim_{x\to 0}\frac{-2\left(\frac{\sin x}{x}\right)^2+2\left(\frac{\sin\frac{3x}{2}}{\frac{3x}{2}}\right)^2\frac{9}{4}}{\left(\frac{\sin 5x}{5x}\right)^2 25}=\frac{1}{10}.
\end{aligned}
$$

c) Određujemo

$$
\begin{aligned}
L &= \lim_{x\to 0}\frac{\tan x-\sin x}{(1+\sin x)\sin^3 x}=\lim_{x\to 0}\frac{\frac{\sin x}{\cos x}-\sin x}{(1+\sin x)\sin^3 x}\\
&=\lim_{x\to 0}\frac{\frac{\sin x(1-\cos x)}{\cos x}}{(1+\sin x)\sin^3 x}=\lim_{x\to 0}\frac{1-\cos x}{(1+\sin x)\sin^2 x\cos x}\\
&=\lim_{x\to 0}\frac{2\sin^2\frac{x}{2}}{(1+\sin x)\sin^2 x\cos x}=\lim_{x\to 0}\frac{2\sin^2\frac{x}{2}}{(1+\sin x)4\sin^2\frac{x}{2}\cos^2\frac{x}{2}\cos x}\\
&=\lim_{x\to 0}\frac{1}{2(1+\sin x)\cos^2\frac{x}{2}\cos x}=\frac{1}{2}.
\end{aligned}
$$

d) Određujemo

$$
\begin{aligned}
L &= \lim_{x\to 0}\frac{1-\cos^3 x}{x\sin 2x}=\lim_{x\to 0}\frac{(1-\cos x)(1+\cos x+\cos^2 x)}{x\sin 2x}\\
&=\lim_{x\to 0}\frac{2\sin^2\frac{x}{2}(1+\cos x+\cos^2 x)}{x\sin 2x}=\lim_{x\to 0}\frac{2\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2\frac{x^2}{4}(1+\cos x+\cos^2 x)}{\frac{\sin 2x}{2x}2x^2}\\
&=\lim_{x\to 0}\frac{\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2(1+\cos x+\cos^2 x)}{4\frac{\sin 2x}{2x}}=\frac{3}{4}
\end{aligned}
$$

e) Određujemo

$$L = \lim_{x\to 0} \frac{1-\cos(1-\cos 2x)}{x^4} = \lim_{x\to 0} \frac{1-\cos(2\sin^2 x)}{x^4} = \lim_{x\to 0} \frac{2\sin^2(\sin^2 x)}{x^4}$$
$$= \lim_{x\to 0} 2\Big(\frac{\sin(\sin^2 x)}{\sin^2 x}\Big)^2 \frac{\sin^4 x}{x^4} = 2.$$

f) Određujemo

$$L = \lim_{x\to 0+} \frac{\sqrt{1-\cos x}}{1-\cos\sqrt{x}} = \lim_{x\to 0+} \frac{\sqrt{2\sin^2\frac{x}{2}}}{2\sin^2\frac{\sqrt{x}}{2}} = \lim_{x\to 0+} \frac{\sqrt{2}\sin\frac{x}{2}}{2\sin^2\frac{\sqrt{x}}{2}}$$
$$= \lim_{x\to 0+} \frac{\sqrt{2}\frac{\sin\frac{x}{2}}{x/2}\frac{x}{2}}{2\Big(\frac{\sin\frac{\sqrt{x}}{2}}{\sqrt{x}/2}\Big)^2\frac{x}{4}} = \sqrt{2}.$$

g) Određujemo

$$L = \lim_{x\to 0} \frac{x}{\sqrt{1+2x+\sin x}-\sqrt{\cos x}}$$
$$= \lim_{x\to 0} \frac{x}{\sqrt{1+2x+\sin x}-\sqrt{\cos x}} \cdot \frac{\sqrt{1+2x+\sin x}+\sqrt{\cos x}}{\sqrt{1+2x+\sin x}+\sqrt{\cos x}}$$
$$= \lim_{x\to 0} \frac{x\Big(\sqrt{1+2x+\sin x}+\sqrt{\cos x}\Big)}{1+2x+\sin x-\cos x} = \lim_{x\to 0} \frac{x\Big(\sqrt{1+2x+\sin x}+\sqrt{\cos x}\Big)}{2\sin^2\frac{x}{2}+2x+\sin x}$$
$$= \lim_{x\to 0} \frac{x\Big(\sqrt{1+2x+\sin x}+\sqrt{\cos x}\Big)}{x\Big(2\frac{\sin^2\frac{x}{2}}{x}+2+\frac{\sin x}{x}\Big)} = \lim_{x\to 0} \frac{\sqrt{1+2x+\sin x}+\sqrt{\cos x}}{\frac{\sin\frac{x}{2}}{\frac{x}{2}}\sin\frac{x}{2}+2+\frac{\sin x}{x}} = \frac{2}{3}.$$

h) Neka je

$$L = \lim_{x\to\pi/4} \tan 2x \tan\Big(\frac{\pi}{4}-x\Big).$$

Uvodimo smenu

$$\frac{\pi}{4}-x = t \quad \Rightarrow \quad x\to\frac{\pi}{4} \quad \Leftrightarrow \quad t\to 0,$$
$$\tan(\frac{\pi}{4}-x) = \tan t,$$
$$\tan 2x = \tan 2(\frac{\pi}{4}-t) = \tan(\frac{\pi}{2}-2t) = \cot 2t = \frac{1}{\tan 2t} = \frac{1-\tan^2 t}{2\tan t}.$$

Sada je

$$L = \lim_{t\to 0} \frac{\tan t(1-\tan^2 t)}{2\tan t} = \lim_{t\to 0} \frac{1-\tan^2 t}{2} = \frac{1}{2}.$$

i) Neka je

$$L = \lim_{x\to \pi/2} \frac{1-\sin x}{\left(\frac{\pi}{2}-x\right)^2}.$$

Uvodimo smenu

$$t = \frac{\pi}{2} - x \quad \Rightarrow \quad x \to \frac{\pi}{2} \quad \Leftrightarrow \quad t \to 0, \qquad \sin x = \sin\left(\frac{\pi}{2}-t\right) = \cos t.$$

Imamo

$$L = \lim_{t\to 0} \frac{1-\cos t}{t^2} = \lim_{t\to 0} \frac{2\sin^2 \frac{t}{2}}{t^2} = \lim_{t\to 0} 2\left(\frac{\sin\frac{t}{2}}{\frac{t}{2}}\right)^2 \frac{1}{4} = \frac{1}{2}.$$

j) Određujemo

$$\begin{aligned} L &= \lim_{x\to 0} \frac{\sqrt{2}-\sqrt{1+\cos x}}{\sin^2 x} = \lim_{x\to 0} \frac{\sqrt{2}-\sqrt{1+\cos x}}{\sin^2 x} \cdot \frac{\sqrt{2}+\sqrt{1+\cos x}}{\sqrt{2}+\sqrt{1+\cos x}} \\ &= \lim_{x\to 0} \frac{1-\cos x}{\sin^2 x(\sqrt{2}+\sqrt{1+\cos x})} = \lim_{x\to 0} \frac{2\sin^2\frac{x}{2}}{4\sin^2\frac{x}{2}\cos^2\frac{x}{2}(\sqrt{2}+\sqrt{1+\cos x})} \\ &= \lim_{x\to 0} \frac{1}{2\cos^2\frac{x}{2}(\sqrt{2}+\sqrt{1+\cos x})} = \frac{1}{4\sqrt{2}}. \end{aligned}$$

7. Odrediti:

a) $\lim\limits_{x\to+\infty}\left(\frac{x^2}{x^2+x+1}\right)^{1+2x}$;

b) $\lim\limits_{x\to 0}(1+\tan x)^{\frac{1}{3x}}$;

c) $\lim\limits_{x\to 0}(2-\cos x)^{\frac{3}{4x^2}}$;

d) $\lim\limits_{x\to 1} x^{\frac{2x}{x-1}}$;

e) $\lim\limits_{x\to 0}(\cos x)^{\cot x/x}$;

f) $\lim\limits_{x\to 0}(x+e^x)^{1/\sin x}$;

g) $\lim\limits_{x\to 0}\left(\frac{\sin x}{x}\right)^{\frac{\sin x}{x-\sin x}}$;

Rešenje: a) Računamo

$$
\begin{aligned}
L &= \lim_{x\to+\infty}\Big(\frac{x^2}{x^2+x+1}\Big)^{1+2x} = \lim_{x\to+\infty}\Big(\frac{x^2+x+1-(x+1)}{x^2+x+1}\Big)^{1+2x} \\
&= \lim_{x\to+\infty}\Big(\Big(1-\frac{x+1}{x^2+x+1}\Big)^{\frac{-(x^2+x+1)}{x+1}}\Big)^{\frac{-(x+1)(1+2x)}{x^2+x+1}}.
\end{aligned}
$$

$$
\begin{aligned}
L_1 &= \lim_{x\to+\infty}\frac{-(x+1)(1+2x)}{x^2+x+1} = \lim_{x\to+\infty}\frac{-x^2\Big(1+\frac{1}{x}\Big)\Big(\frac{1}{x}+2\Big)}{x^2\Big(1+\frac{1}{x}+\frac{1}{x^2}\Big)} \\
&= \lim_{x\to+\infty}\frac{-\Big(1+\frac{1}{x}\Big)\Big(\frac{1}{x}+2\Big)}{\Big(1+\frac{1}{x}+\frac{1}{x^2}\Big)} = -2.
\end{aligned}
$$

Sada je

$$L = e^{-2}.$$

b) Nalazimo

$$
L = \lim_{x\to0}(1+\tan x)^{\frac{1}{3x}} = \lim_{x\to0}\Big((1+\tan x)^{\frac{1}{\tan x}}\Big)^{\frac{\tan x}{3x}} = \lim_{x\to0}\Big((1+\tan x)^{\frac{1}{\tan x}}\Big)^{\frac{\sin x}{x}\cdot\frac{1}{3\cos x}}
$$
$$= e^{1/3}.$$

c) Neka je

$$
L = \lim_{x\to0}(2-\cos x)^{\frac{3}{4x^2}} = \lim_{x\to0}\Big((1+1-\cos x)^{\frac{1}{1-\cos x}}\Big)^{\frac{3(1-\cos x)}{4x^2}}. \tag{0.1}
$$

Imamo

$$
\lim_{x\to0}\frac{3(1-\cos x)}{4x^2} = \lim_{x\to0}\frac{6\sin^2\frac{x}{2}}{4x^2} = \lim_{x\to0}\frac{3}{8}\Big(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\Big)^2 = \frac{3}{8}. \tag{0.2}
$$

Na osnovu (0.1) i (0.2) dobijamo

$$L = e^{3/8}.$$

d) Određujemo

$$
L = \lim_{x\to1}x^{\frac{2x}{x-1}} = \lim_{x\to1}(1+x-1)^{\frac{2x}{x-1}} = \lim_{x\to1}\Big((1+x-1)^{\frac{1}{x-1}}\Big)^{2x} = e^2.
$$

e) Računamo

$$L = \lim_{x\to 0}(\cos x)^{\cot x/x} = \lim_{x\to 0}\Big((1+\cos x-1)^{1/(\cos x-1)}\Big)^{\frac{(\cos x-1)\cot x}{x}}. \tag{0.3}$$

Imamo

$$\begin{aligned}\lim_{x\to 0}\frac{(\cos x-1)\cot x}{x} &= \lim_{x\to 0}\frac{(\cos x-1)\cos x}{x\sin x} = \lim_{x\to 0}\frac{-2\sin^2\frac{x}{2}\cos x}{x\sin x}\\ &= \lim_{x\to 0}\frac{-2\frac{\sin^2(x/2)}{(x/2)^2}\frac{1}{4}\cos x}{\frac{\sin x}{x}} = -\frac{1}{2}.\end{aligned} \tag{0.4}$$

Zamenom (0.4) u (0.3) dobijamo $L = e^{-1/2}$.

f) Određujemo

$$L = \lim_{x\to 0}(x+e^x)^{1/\sin x} = \lim_{x\to 0}\Big((1+x+e^x-1)^{1/(x+e^x-1)}\Big)^{(x+e^x-1)/\sin x}. \tag{0.5}$$

Imamo

$$\begin{aligned}\lim_{x\to 0}\frac{x+e^x-1}{\sin x} &= \lim_{x\to 0}\Big(\frac{x}{\sin x}+\frac{e^x-1}{\sin x}\Big)\\ = \lim_{x\to 0}\Big(\frac{x}{\sin x}+\frac{\frac{e^x-1}{x}}{\frac{\sin x}{x}}\Big) &= 1+1 = 2.\end{aligned} \tag{0.6}$$

Zamenom (0.6) u (0.5) dobijamo

$$L = e^2.$$

g) Određujemo

$$\begin{aligned}L &= \lim_{x\to 0}\Big(\frac{\sin x}{x}\Big)^{\frac{\sin x}{x-\sin x}} = \lim_{x\to 0}\Big(1+\frac{\sin x}{x}-1\Big)^{\frac{\sin x}{x-\sin x}}\\ &= \lim_{x\to 0}\bigg(\Big(1+\frac{\sin x - x}{x}\Big)^{\frac{x}{\sin x - x}}\bigg)^{\frac{\sin x}{-x}} = e^{-1}.\end{aligned}$$