II Метод Остроградског

$$I = \int \frac{P_n(x)\,dx}{\sqrt{ax^2+bx+c}} = Q_{n-1}(x)\sqrt{ax^2+bx+c} + \lambda \int \frac{dx}{\sqrt{ax^2+bx+c}} \quad (*)$$

$a, b, c \in \mathbb{R}$

$P_n(x)$ – полином степена n

$Q_{n-1}(x)$ – полином степена $n-1$ чије коефицијенте треба одредити

λ – непознати коефицијент

Коефицијенти полинома Q_{n-1} и коефицијент λ одређују се диференцирањем $(*)$.

1. $I = \int \frac{x^2\,dx}{\sqrt{x^2-x+1}}$

$P_n(x) = P_2(x) = x^2$, $n = 2$; $Q_{n-1}(x) = Q_1(x) = Ax+B$

$$\cancel{\int \frac{x^2dx}{\sqrt{x^2-x+1}} = (Ax+B)\sqrt{x^2-x+1} + \lambda \int \frac{dx}{\sqrt{x^2-x+1}} \Big/'_x}$$

$$\int \frac{x^2dx}{\sqrt{x^2-x+1}} = (Ax+B)\sqrt{x^2-x+1} + \lambda \int \frac{dx}{\sqrt{x^2-x+1}} \Big/'_x$$

$$\frac{x^2}{\sqrt{x^2-x+1}} = A\sqrt{x^2-x+1} + (Ax+B)\frac{2x-1}{2\sqrt{x^2-x+1}} + \lambda \cdot \frac{1}{\sqrt{x^2-x+1}} \quad \Big/ \cdot 2\sqrt{x^2-x+1}$$

$$2x^2 = 2A(x^2-x+1) + (Ax+B)(2x-1) + 2\lambda$$

$$2x^2 = 2Ax^2 - 2Ax + 2A + 2Ax^2 - Ax + 2Bx - B + 2\lambda$$

$$2x^2 = 4Ax^2 + (-3A+2B)x + 2A - B + 2\lambda$$

$x^2: \quad 2 = 4A \quad \Rightarrow A = 1/2$

$x: \quad 0 = -3A + 2B \quad \Rightarrow B = 3/4$

$1: \quad 0 = 2A - B + 2\lambda \quad \Rightarrow \lambda = \frac{1}{2}(B - 2A) = -1/8$

$$I = \left(\frac{1}{2}x + \frac{3}{4}\right)\sqrt{x^2-x+1} - \frac{1}{8}\underbrace{\int \frac{dx}{\sqrt{x^2-x+1}}}_{I_1}$$

$$x^2 - x + 1 = \left(x - \frac{1}{2}\right)^2 + \frac{3}{4} = \frac{3}{4}\left(\left(\frac{\cancel{2}}{\sqrt{3}}\frac{2x-1}{\cancel{2}}\right)^2 + 1\right)$$

$$I_1 = \int \frac{dx}{\sqrt{\frac{3}{4}\left(\left(\frac{2x-1}{\sqrt{3}}\right)^2+1\right)}} = \left\{\begin{array}{l} \frac{2x-1}{\sqrt{3}} = t \\ \frac{2}{\sqrt{3}}dx = dt \end{array}\right\} = \int \frac{\frac{\sqrt{3}}{\cancel{2}}dt}{\frac{\sqrt{3}}{\cancel{2}}\sqrt{t^2+1}} =$$

$$= \left\{\begin{array}{l} t = \operatorname{sh} s \\ dt = \operatorname{ch} s\, ds \end{array}\right\} = \int \frac{\operatorname{ch} s\, ds}{\sqrt{\operatorname{sh}^2 s + 1}} = \int ds + C = s + C =$$

$$= \operatorname{arsh} t + C = \ln\left(t + \sqrt{t^2+1}\right) + C =$$

$$= \ln\left(\frac{2x-1}{\sqrt{3}} + \sqrt{\underbrace{\frac{(2x-1)^2}{3} + 1}_{\frac{4x^2-4x+1+3}{3} = \frac{4(x^2-x+1)}{3}}}\right) + C$$

$$= \ln\left(\frac{1}{\sqrt{3}}\left(2x - 1 + 2\sqrt{x^2-x+1}\right)\right) + C$$

$$I = \frac{1}{2}x + \frac{3}{4}\sqrt{x^2-x+1} + \ln\frac{1}{\sqrt{3}}\left(2x-1+2\sqrt{x^2-x+1}\right) + C$$

2. $I = \int \frac{x^3+1}{\sqrt{1-x-x^2}}\,dx$

$P_n(x) = P_3(x) = x^3 + 1, \quad n = 3; \quad Q_{n-1}(x) = Q_2(x) = Ax^2 + Bx + C$

$$\int \frac{x^3+1}{\sqrt{1-x-x^2}}\,dx = (Ax^2 + Bx + C)\sqrt{1-x-x^2} + \lambda \int \frac{dx}{\sqrt{1-x-x^2}} \Big|'_x$$

$$\frac{x^3+1}{\sqrt{1-x-x^2}} = (2Ax+B)\sqrt{1-x-x^2} + (Ax^2+Bx+C)\frac{-1-2x}{2\sqrt{1-x-x^2}} + \lambda\frac{1}{\sqrt{1-x-x^2}} \quad \Big| \cdot 2\sqrt{1-x-x^2}$$

$$2x^3+2 = (4Ax+2B)(1-x-x^2) + (Ax^2+Bx+C)(-2x-1) + 2\lambda$$

$$2x^3+2 = 4Ax - 4Ax^2 - 4Ax^3 + 2B - 2Bx - 2Bx^2$$
$$-2Ax^3 - Ax^2 - 2Bx^2 - Bx - 2Cx - C + 2\lambda$$

$x^3:\ 2 = -4A-2A$	$2=-6A$
$x^2:\ 0 = -4A-2B-A-2B$	$0=-5A-4B$
$x:\ 0 = 4A-2B-B-2C$	$0=4A-3B-2C$
$1:\ 2 = 2B-C+2\lambda$	$2=2B-C+2\lambda$

$A = -1/3$

$B = -5\cdot(-1/3)\,/\,4 = \frac{5}{12}$

$C = \left(4\cdot\left(-\frac{1}{3}\right) - 3\cdot\frac{5}{12}\right)/2 = \left(-\frac{4}{3}-\frac{5}{4}\right)/2 = \frac{-16-15}{12\cdot 2} = -\frac{31}{24}$

$\lambda = \left(2-\frac{5}{6}-\frac{31}{24}\right)/2 = \frac{48-20-31}{24\cdot 2} = \frac{-\cancel{3}}{\cancel{24}_{8}\cdot 2} = -\frac{1}{16}$

$$I = \left(-\frac{1}{3}x^2+\frac{5}{12}x-\frac{31}{24}\right)\sqrt{1-x-x^2} - \frac{1}{16}\underbrace{\int\frac{dx}{\sqrt{1-x-x^2}}}_{I_1}$$

$1-x-x^2 = -(x^2+x-1) = -\left(\left(x+\frac{1}{2}\right)^2-\frac{5}{4}\right) =$
$= -\frac{5}{4}\left(\left(\frac{2}{\sqrt5}\cdot\frac{2x+1}{\cancel{2}}\right)^2-1\right)$

$$I_1 = \int\frac{dx}{\sqrt{\frac{5}{4}\left(\left(\frac{2x+1}{\sqrt5}\right)^2-1\right)}} = \int\frac{dx}{\frac{\sqrt5}{2}\sqrt{1-\left(\frac{2x+1}{\sqrt5}\right)^2}} = \left|\begin{array}{l}\frac{2x+1}{\sqrt5}=t\\ \frac{2dx}{\sqrt5}=dt\end{array}\right|$$

$$= \int\frac{\frac{\cancel{\sqrt5}}{\cancel{2}}dt}{\frac{\cancel{\sqrt5}}{\cancel{2}}\sqrt{1-t^2}} = \arcsin t + C = \arcsin\frac{2x+1}{\sqrt5}+C$$

$$I = \left(-\frac{1}{3}x^2+\frac{5}{12}x-\frac{31}{24}\right)\sqrt{1-x-x^2} - \frac{1}{16}\arcsin\frac{2x+1}{\sqrt5}+C$$