Рекурентне формуле

1. (а) Одредити рекурентну ф-лу за $I_n = \int \ln^n x\,dx$,

(б) Одредити $I_4 = \int \ln^4 x\,dx$.

$$I_n = \int \ln^n x\,dx = \left\{ \begin{array}{l} u = \ln^n x, \quad dv = dx \\ du = n\ln^{n-1}x \cdot \frac{1}{x}dx, \; v = x \end{array} \right\} = x\ln^n x - n\int \ln^{n-1} x\,dx =$$

$$= x\ln^n x - nI_{n-1}$$

$I_1 = \int \ln x\,dx = \dots = x\ln x - x + C$ (иницијални)

$I_n = x\ln^n x - nI_{n-1}, \quad I_1 = x\ln x - x + C$ — рекурентна ф-ла

$$I_4 = x\ln^4 x - 4I_3 = x\ln^4 x - 4(x\ln^3 x - 3I_2) =$$

$$= x\ln^4 x - 4(x\ln^3 x - 3(x\ln^2 x - 2I_1)) =$$

$$= x\ln^4 x - 4(x\ln^3 x - 3(x\ln^2 x - 2(x\ln x - x))) + C$$

2. (а) $I_n = \int x^n e^x\,dx$, (б) $I_5 = \int x^5 e^x\,dx$

$$I_n = \int x^n e^x\,dx = \left\{ \begin{array}{l} u = x^n, \quad dv = e^x dx \\ du = nx^{n-1}dx, \; v = e^x \end{array} \right\} = x^n e^x - n\int x^{n-1} e^x\,dx$$

$$= x^n e^x - nI_{n-1}$$

$I_0 = \int e^x dx = e^x + C$

$I_n = x^n e^x - nI_{n-1}, \quad I_0 = e^x + C$ — рекурентна ф-ла

$$I_5 = x^5 e^x - 5I_4 = x^5 e^x - 5(x^4 e^x - 4I_3) = x^5 e^x - 5(x^4 e^x - 4(x^3 e^x - 3I_2)) =$$

$$= x^5 e^x - 5(x^4 e^x - 4(x^3 e^x - 3(x^2 e^x - 2I_1))) =$$

$$= x^5 e^x - 5(x^4 e^x - 4(x^3 e^x - 3(x^2 e^x - 2(xe^x - I_0)))) =$$

$$= x^5 e^x - 5(x^4 e^x - 4(x^3 e^x - 3(x^2 e^x - 2(xe^x - e^x)))) + C$$

3. (а) $I_n = \int \frac{dx}{(x^2+1)^n}$, (б) $I_3 = \int \frac{dx}{(x^2+1)^3}$

$$I_n = \int \frac{dx}{(x^2+1)^n} = \int \frac{(x^2+1) - x^2}{(x^2+1)^n}\,dx = \int \frac{dx}{(x^2+1)^{n-1}} - \int x\,\frac{x\,dx}{(x^2+1)^n} =$$

$$= \left\{ \begin{array}{l} u = x, \quad d\vartheta = \frac{x\,dx}{(x^2+1)^n} \\ du = dx, \quad \vartheta = \int \frac{x\,dx}{(x^2+1)^n} = \left\{ \begin{array}{l} x^2+1 = t \\ 2x\,dx = dt \end{array} \right\} = \int \frac{dt/2}{t^n} = \frac{1}{2}\,\frac{t^{1-n}}{1-n} = \frac{(x^2+1)^{1-n}}{2(1-n)} \end{array} \right\} =$$

$$= \int \frac{dx}{(x^2+1)^{n-1}} - \frac{1}{2}\,\frac{x}{(1-n)(x^2+1)^{n-1}} + \frac{1}{2(1-n)} \int \frac{dx}{(x^2+1)^{n-1}} =$$

$$= \frac{2-2n+1}{2(1-n)} \int \frac{dx}{(x^2+1)^{n-1}} + \frac{x}{2(n-1)(x^2+1)^{n-1}} =$$

$$= \frac{x}{2(n-1)(x^2+1)^{n-1}} + \frac{2n-3}{2(n-1)}\,I_{n-1}$$

$I_1 = \int \frac{dx}{x^2+1} = \operatorname{arctg} x + C$ (иницијални)

$I_n = \frac{x}{2(n-1)(x^2+1)^{n-1}} + \frac{2n-3}{2(n-1)}\,I_{n-1}$, $I_1 = \operatorname{arctg} x + C$ – рек. ф-ла.

$$I_3 = \frac{x}{2\cdot 2\,(x^2+1)^2} + \frac{3}{2\cdot 2}\,I_2 = \frac{x}{4(x^2+1)^2} + \frac{3}{4}\left(\frac{x}{2\cdot 1\cdot (x^2+1)^1} + \frac{1}{2\cdot 1}\,I_1\right)$$

$$= \frac{x}{4(x^2+1)^2} + \frac{3x}{8(x^2+1)} + \frac{3}{8}\operatorname{arctg} x + C$$

4. (а) $I_n = \int \sin^n x\,dx$, (б) $I_7 = \int \sin^7 x\,dx$, (в) $I_8 = \int \sin^8 x\,dx$

$$I_n = \int \sin^n x\,dx = \int \sin^{n-1}x \cdot \sin x\,dx = \left\{ \begin{array}{l} u = \sin^{n-1}x, \quad dv = \sin x\,dx \\ du = (n-1)\sin^{n-2}x \cdot \cos x\,dx, \quad v = -\cos x \end{array} \right\} =$$

$$= -\sin^{n-1}x\cos x + (n-1)\int \sin^{n-2}x \underbrace{\cos^2 x}_{1-\sin^2 x}\,dx =$$

$$= -\sin^{n-1}x\cos x + (n-1)\underbrace{\int \sin^{n-2}x\,dx}_{I_{n-2}} - (n-1)\underbrace{\int \sin^n x\,dx}_{I_n}$$

$$n I_n = -\sin^{n-1}x\cos x + (n-1) I_{n-2}$$

$$I_n = -\frac{1}{n}\sin^{n-1}x\cos x + \cancel{(n-1)}\ \frac{n-1}{n} I_{n-2}$$

$$I_1 \Rightarrow I_3 \Rightarrow I_5 \Rightarrow \dots, \quad I_2 \Rightarrow I_4 \Rightarrow I_6 \Rightarrow \dots$$

(иницијални)

$$\left| \begin{array}{l} I_1 = \int \sin x\,dx = -\cos x + C \\ I_2 = \int \sin^2 x\,dx = \int \frac{1-\cos 2x}{2}\,dx = \frac{1}{2}\left(x - \frac{1}{2}\sin 2x\right) + C = \frac{1}{2}x - \frac{1}{4}\sin 2x + C \end{array} \right.$$

$$\left. \begin{array}{l} I_n = -\frac{1}{n}\sin^{n-1}x\cos x + \frac{n-1}{n} I_{n-2}, \\ I_1 = -\cos x + C, \quad I_2 = \frac{1}{2}x - \frac{1}{4}\sin 2x + C \end{array} \right\}$$ рекурентна ф-ла

$$I_7 = -\frac{1}{7}\sin^6 x\cos x + \frac{6}{7} I_5 = -\frac{1}{7}\sin^6 x\cos x + \frac{6}{7}\left(-\frac{1}{5}\sin^4 x\cos x + \frac{4}{5} I_3\right) =$$

$$= -\frac{1}{7}\sin^6 x\cos x + \frac{6}{7}\left(-\frac{1}{5}\sin^4 x\cos x + \frac{4}{5}\left(-\frac{1}{3}\sin^2 x\cos x + \frac{2}{3} I_1\right)\right) =$$

$$= -\frac{1}{7}\sin^6 x\cos x + \frac{6}{7}\left(-\frac{1}{5}\sin^4 x\cos x + \frac{4}{5}\left(-\frac{1}{3}\sin^2 x\cos x + \frac{2}{3}(-\cos x)\right)\right) + C$$

$$I_8 = -\frac{1}{8}\sin^7 x\cos x + \frac{7}{8} I_6 = -\frac{1}{8}\sin^7 x\cos x + \frac{7}{8}\left(-\frac{1}{6}\sin^5 x\cos x + \frac{5}{6} I_4\right) =$$

$$= -\frac{1}{8}\sin^7 x\cos x + \frac{7}{8}\left(-\frac{1}{6}\sin^5 x\cos x + \frac{5}{6}\left(-\frac{1}{4}\sin^3 x\cos x + \frac{3}{4} I_2\right)\right) =$$

$$= -\frac{1}{8}\sin^7 x\cos x + \frac{7}{8}\left(-\frac{1}{6}\sin^5 x\cos x + \frac{5}{6}\left(-\frac{1}{4}\sin^3 x\cos x + \frac{3}{4}\left(\frac{1}{2}x - \frac{1}{4}\sin 2x\right)\right)\right) + C$$

5. (а) $I_n = \int \cos^n x\,dx$, (б) $I_5 = \int \cos^5 x\,dx$, (в) $I_6 = \int \cos^6 x\,dx$

$$I_n = \int \cos^n x\,dx = \int \cos^{n-1}x \cos x\,dx = \left\{ \begin{array}{ll} u = \cos^{n-1}x, & dv = \cos x\,dx \\ du = (n-1)\cos^{n-2}x\,(-\sin x), & v = \sin x \end{array} \right\} =$$

$$= \cos^{n-1}x \sin x + (n-1)\int \cos^{n-2}x \underbrace{\sin^2 x}_{1-\cos^2 x}\,dx =$$

$$= \cos^{n-1}x \sin x + (n-1)\underbrace{\int \cos^{n-2}x\,dx}_{I_{n-2}} \underbrace{+}_{(n-1)} \underbrace{\int \cos^n x\,dx}_{I_n}$$

$$n I_n = \cos^{n-1}x \sin x + (n-1) I_{n-2}$$

$$I_n = \frac{1}{n}\cos^{n-1}x \sin x + \frac{n-1}{n} I_{n-2}$$

(ініціалізація)

$$\begin{cases} I_1 = \int \cos x\,dx = \sin x + C \\ I_2 = \int \cos^2 x\,dx = \int \frac{1+\cos 2x}{2} = \frac{1}{2}x + \frac{1}{4}\sin 2x + C \end{cases}$$

$$\left. \begin{array}{l} I_n = \frac{1}{n}\cos^{n-1}x \sin x + \frac{n-1}{n} I_{n-2} \\ I_1 = \sin x + C, \quad I_2 = \frac{1}{2}x + \frac{1}{4}\sin 2x + C \end{array} \right\} \text{рекурентна ф-ула}$$

$$I_5 = \frac{1}{5}\cos^4 x \sin x + \frac{4}{5} I_3 = \frac{1}{5}\cos^4 x \sin x + \frac{4}{5}\left(\frac{1}{3}\cos^2 x \sin x + \frac{2}{3} I_1\right) =$$

$$= \frac{1}{5}\cos^4 x \sin x + \frac{4}{5}\left(\frac{1}{3}\cos^2 x \sin x + \frac{2}{3}\sin x\right) + C$$

$$I_6 = \frac{1}{6}\cos^5 x \sin x + \frac{5}{6} I_4 = \frac{1}{6}\cos^5 x \sin x + \frac{5}{6}\left(\frac{1}{4}\cos^3 x \sin x + \frac{3}{4} I_2\right) =$$

$$= \frac{1}{6}\cos^5 x \sin x + \frac{5}{6}\left(\frac{1}{4}\cos^3 x \sin x + \frac{3}{4}\left(\frac{1}{2}x + \frac{1}{4}\sin 2x\right)\right) + C$$