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Poznati limesi

$$\lim_{x\to 0}\frac{\sin x}{x}=1 \qquad \lim_{x\to 0}\frac{\operatorname{sh} x}{x}=1 \qquad \lim_{x\to 0}\frac{1-\cos x}{x^2}=\frac{1}{2} \qquad \lim_{x\to 0}\frac{\operatorname{ch} x-1}{x^2}=\frac{1}{2}$$

$$\lim_{x\to 0}(1+x)^{\frac{1}{x}}=e \qquad \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x=e \qquad \lim_{x\to 0}\frac{e^x-1}{x}=1 \qquad \lim_{x\to 0}\frac{\ln(1+x)}{x}=1$$

1. $$\lim_{x\to\infty}\frac{2x+1}{3x-2}\overset{\frac{\infty}{\infty}}{=}\lim_{x\to\infty}\frac{\cancel{x}\cdot\left(2+\frac{1}{x}\right)}{\cancel{x}\cdot\left(3-\frac{2}{x}\right)}=\lim_{x\to\infty}\frac{2+\cancel{\frac{1}{x}}^{\to 0}}{3-\cancel{\frac{2}{x}}^{\to 0}}=\frac{2}{3}$$

2. $$\lim_{x\to\infty}\frac{x^2-3x+1}{2x^2+x}\overset{\frac{\infty}{\infty}}{=}\lim_{x\to\infty}\frac{\cancel{x^2}\left(1-\frac{3}{x}+\frac{1}{x^2}\right)}{\cancel{x^2}\left(2+\frac{1}{x}\right)}=\lim_{x\to\infty}\frac{1-\cancel{\frac{3}{x}}^{\to 0}+\cancel{\frac{1}{x^2}}^{\to 0}}{2+\cancel{\frac{1}{x}}^{\to 0}}=\frac{1}{2}$$

3. $$\lim_{x\to\infty}\frac{5x^2-x+3}{3x^3+2x-4}\overset{\frac{\infty}{\infty}}{=}\lim_{x\to\infty}\frac{\cancel{x^3}\left(\cancel{\frac{5}{x}}^{\to 0}-\cancel{\frac{1}{x^2}}^{\to 0}+\cancel{\frac{3}{x^3}}^{\to 0}\right)}{\cancel{x^3}\left(3+\cancel{\frac{2}{x^2}}^{\to 0}-\cancel{\frac{4}{x^3}}^{\to 0}\right)}=\frac{0}{3}=0$$

4. $$\lim_{x\to\infty}\frac{6x^4-2x^3+x^2}{2x^3+x^2-3}=\lim_{x\to\infty}\frac{\cancel{x^4}\left(6-\cancel{\frac{2}{x}}^{\to 0}+\cancel{\frac{1}{x^2}}^{\to 0}\right)}{\cancel{x^4}\left(\cancel{\frac{2}{x}}^{\to 0}+\cancel{\frac{1}{x^2}}^{\to 0}-\cancel{\frac{3}{x^4}}^{\to 0}\right)}=\frac{6}{0}=+\infty$$

5. a) $$\lim_{x\to+\infty}\frac{\sqrt{x^2-2x}}{x+1}=\lim_{x\to+\infty}\frac{\sqrt{x^2\left(1+\frac{2}{x}\right)}}{x\cdot\left(1+\frac{1}{x}\right)}=\lim_{x\to\infty}\frac{|x|\cdot\sqrt{1+\frac{2}{x}}}{x\cdot\left(1+\frac{1}{x}\right)}\overset{*}{=}\lim_{x\to\infty}\frac{\cancel{x}\cdot\sqrt{1+\cancel{\frac{2}{x}}^{\to 0}}}{\cancel{x}\cdot\left(1+\cancel{\frac{1}{x}}^{\to 0}\right)}=1$$

$^*\ |x|=x$ jer $x\to+\infty$

b) $$\lim_{x\to-\infty}\frac{\sqrt{x^2-2x}}{x+1}=\lim_{x\to-\infty}\frac{|x|\cdot\sqrt{1+\frac{2}{x}}}{x\cdot\left(1+\frac{1}{x}\right)}\overset{*}{=}\lim_{x\to-\infty}\frac{-\cancel{x}\cdot\sqrt{1+\cancel{\frac{2}{x}}^{\to 0}}}{\cancel{x}\cdot\left(1+\cancel{\frac{1}{x}}^{\to 0}\right)}=-1 \qquad !!!$$

$|x|=-x$ jer $x\to-\infty$

6. a) $$\lim_{x\to\infty}\frac{x-\sqrt{x^2+3x}}{2x+1}=\lim_{x\to\infty}\frac{x-\sqrt{x^2\left(1+\frac{3}{x}\right)}}{2x+1}=\lim_{x\to\infty}\frac{x-x\cdot\sqrt{1+\frac{3}{x}}}{2x+1}=$$

$$\lim_{x\to\infty}\frac{\cancel{x}\cdot\left(1-\sqrt{1+\cancelto{0}{\frac{3}{x}}}\right)}{\cancel{x}\cdot\left(2+\cancelto{0}{\frac{1}{x}}\right)}=\frac{1-1}{2}=0$$

b) $$\lim_{x\to-\infty}\frac{x-\sqrt{x^2+3x}}{2x+1}=\lim_{x\to-\infty}\frac{x-\sqrt{x^2\left(1+\frac{3}{x}\right)}}{2x+1}=\lim_{x\to-\infty}\frac{x-(-x)\cdot\sqrt{1+\frac{3}{x}}}{2x+1}=$$

$$\lim_{x\to-\infty}\frac{\cancel{x}\cdot\left(1+\sqrt{1+\cancelto{0}{\frac{3}{x}}}\right)}{\cancel{x}\left(2+\cancelto{0}{\frac{1}{x}}\right)}=\frac{1+1}{2}=1$$

Zadaci za vezbu:

1. $$\lim_{x\to\infty}\frac{2x-1+\sqrt{x^2-x}}{3x+\sqrt{x^2+7}}=\frac{3}{4}$$

2. $$\lim_{x\to-\infty}\frac{2x-1+\sqrt{x^2-x}}{3x+\sqrt{x^2+7}}=\frac{1}{2}$$

3. $$\lim_{x\to\infty}\frac{\sqrt[3]{x^3-2x}+4x-1}{2\sqrt{x^2+3x}+x}=\frac{5}{3}$$

4. $$\lim_{x\to-\infty}\frac{4\cdot\sqrt{x^2-5x+1}+\sqrt[3]{x^3+x^2-7}}{3x+4+\sqrt{x^2-2x+5}}=\frac{-3}{2}$$

Neodređeni oblik „$\infty-\infty$"

1. $$\lim_{x\to\infty}\left(\sqrt{x}-\sqrt{x-3}\right) \overset{\text{"}\infty-\infty\text{"}}{=} \lim_{x\to\infty}\left(\sqrt{x}-\sqrt{x-3}\right)\cdot\frac{\sqrt{x}+\sqrt{x-3}}{\sqrt{x}+\sqrt{x-3}} = \lim_{x\to\infty}\frac{x-(x-3)}{\sqrt{x}+\sqrt{x-3}} =$$

$$= \lim_{x\to\infty}\frac{3}{\sqrt{x}+\sqrt{x-3}} = 0$$

2. $$\lim_{x\to\infty}\left(x-\sqrt{x^2-3x+4}\right) = \lim_{x\to\infty}\left(x-\sqrt{x^2-3x+4}\right)\cdot\frac{x+\sqrt{x^2-3x+4}}{x+\sqrt{x^2-3x+4}} = \lim_{x\to\infty}\frac{x^2-(x^2-3x+4)}{x+\sqrt{x^2-3x+4}} =$$

$$= \lim_{x\to\infty}\frac{3x-4}{x+\sqrt{x^2-3x+4}} = \lim_{x\to\infty}\frac{\cancel{x}\cdot\left(3-\frac{4}{x}\right)}{\cancel{x}\cdot\left(1+\sqrt{1-\frac{3}{x}+\frac{4}{x^2}}\right)} = \frac{3}{2}$$

3. $$\lim_{x\to-\infty}\left(x+\sqrt{x^2-x+2}\right) = \lim_{x\to-\infty}\left(x+\sqrt{x^2-x+2}\right)\cdot\frac{x-\sqrt{x^2-x+2}}{x-\sqrt{x^2-x+2}} = \lim_{x\to\infty}\frac{x^2-(x^2-x+2)}{x-\sqrt{x^2-x+2}} =$$

$$= \lim_{x\to-\infty}\frac{x+2}{x-\sqrt{x^2-x+2}} = \lim_{x\to-\infty}\frac{\cancel{x}\left(1+\frac{2}{x}\right)}{\cancel{x}\cdot\left(1+\sqrt{1-\frac{1}{x}+\frac{2}{x^2}}\right)} = \frac{1}{2}$$

4. $$\lim_{x\to\infty}\left(x-\sqrt[3]{x^3+2x^2-1}\right) = \lim_{x\to\infty}\left(x-\sqrt[3]{x^3+2x^2-1}\right)\cdot\frac{x^2+x\cdot\sqrt[3]{x^3+2x^2-1}+\sqrt[3]{(x^3+2x^2-1)^2}}{x^2+x\cdot\sqrt[3]{x^3+2x^2-1}+\sqrt[3]{(x^3+2x^2-1)^2}} =$$

$$= \lim_{x\to\infty}\frac{x^3-(x^3+2x^2-1)}{x^2+x\cdot\sqrt[3]{x^3+2x^2-1}+\sqrt[3]{(x^3+2x^2-1)^2}} = \lim_{x\to\infty}\frac{-2x^2+1}{x^2+x\cdot\sqrt[3]{x^3+2x^2-1}+\sqrt[3]{(x^3+2x^2-1)^2}} = \frac{-2}{1+1+1} =$$

$$\boxed{\sqrt[3]{x^3}=x} \qquad = -\frac{2}{3}$$

Zadaci za vežbu:

1. $\lim_{x\to-\infty}\left(x+\sqrt{x^2-3x+1}\right) = \frac{3}{2}$

2. $\lim_{x\to\infty}\left(\sqrt{x^2-3x+1}-x\right) = -\frac{3}{2}$

3. $\lim_{x\to\infty}\left(\sqrt[4]{x^4+x^3-2}-\sqrt[4]{x^4-x^2+3x}\right) = \frac{1}{4}$

Primena tablicnih limesa

1. $$\lim_{x\to 0}(1-\cos x)\cdot \operatorname{ctg} x \overset{\frac{0}{0}}{=} \lim_{x\to 0}\frac{1-\cos x}{x^2}\cdot x^2\cdot \operatorname{ctg} x = \lim_{x\to 0}\frac{1-\cos x}{x^2}\cdot x^2\cdot\frac{\cos x}{\sin x} =$$

$$= \lim_{x\to 0}\underbrace{\frac{1-\cos x}{x^2}}_{\to\frac{1}{2}}\cdot\underbrace{\frac{x}{\sin x}}_{\to 1}\cdot x\cdot\cos x = 0$$

2. $$\lim_{x\to 0}\frac{\cos 2x-\cos 3x}{\sin^2 5x}$$

I način:

$$\lim_{x\to 0}\frac{\cos 2x-\cos 3x}{\sin^2 5x} = \lim_{x\to 0}\frac{2\sin\frac{5x}{2}\sin\frac{x}{2}}{\sin^2 5x} = \lim_{x\to 0}\frac{2\cdot\underbrace{\frac{\sin\frac{5x}{2}}{\frac{5x}{2}}}_{\to 1}\cdot\underbrace{\frac{\sin\frac{x}{2}}{\frac{x}{2}}}_{\to 1}\cdot\frac{5x}{2}\cdot\frac{x}{2}}{\underbrace{\left(\frac{\sin 5x}{5x}\right)^2}_{\to 1}\cdot 25x^2} = \frac{-1}{10}$$

II način:

$$\lim_{x\to 0}\frac{\cos 2x-\cos 3x}{\sin^2 5x} = \lim_{x\to 0}\frac{(\cos 2x-1)+(1-\cos 3x)}{\sin^2 5x} = \lim_{x\to 0}\frac{\underbrace{\frac{\cos 2x-1}{(2x)^2}}_{\to -\frac{1}{2}}\cdot(2x)^2+\underbrace{\frac{1-\cos 3x}{(3x)^2}}_{\to\frac{1}{2}}\cdot(3x)^2}{\underbrace{\left(\frac{\sin 5x}{5x}\right)^2}_{\to 1}\cdot 25x^2} =$$

$$= \lim_{x\to 0}\frac{-\frac{1}{2}\cdot 4x^2+\frac{1}{2}\cdot 9x^2}{25x^2} = \lim_{x\to 0}\frac{-2x^2+\frac{9}{2}x^2}{25x^2} = \frac{1}{10}$$

3. $$\lim_{x\to 0}\frac{1-\cos^3 x}{x\sin 2x}=\lim_{x\to 0}\frac{(1-\cos x)(1+\cos x+\cos^2 x)}{x\cdot\sin 2x}=$$

$$\lim_{x\to 0}\frac{\underbrace{\frac{1-\cos x}{x^2}}_{\to\frac{1}{2}}\cdot x^2\cdot\overbrace{(1+\cos x+\cos^2 x)}^{3}}{x\cdot\underbrace{\frac{\sin 2x}{2x}}_{\to 1}\cdot 2x}=\frac{3}{4}$$

4. $$\lim_{x\to 0^+}\frac{\sqrt{1-\cos x}}{1-\cos\sqrt{x}}=\lim_{x\to 0^+}\frac{\sqrt{\underbrace{\frac{1-\cos x}{x^2}}_{\to\frac{1}{2}}\cdot x^2}}{\underbrace{\frac{1-\cos\sqrt{x}}{(\sqrt{x})^2}}_{\to\frac{1}{2}}\cdot x}=\lim_{x\to 0^+}\frac{\frac{x}{\sqrt{2}}}{\frac{x}{2}}=\sqrt{2}$$

Za vežbu:

$$\lim_{x\to 0}\frac{1-\cos(1-\cos 2x)}{x^4}=2$$

$$\lim_{x\to\frac{\pi}{2}}\frac{1-\sin x}{(\frac{\pi}{2}-x)^2}=\frac{1}{2}$$

$$\lim_{x\to 0}\frac{\sin(1-\cos(2\sqrt{2}x))}{x^2}=4$$

$$\lim_{x\to 0}\frac{\sqrt{2}-\sqrt{1+\cos x}}{\sin^2 x}=\frac{\sqrt{2}}{8}$$

$$\lim_{x\to\frac{\pi}{4}}\operatorname{tg}2x\cdot\operatorname{tg}\left(\frac{\pi}{4}-x\right)=\frac{1}{2}$$

1. $$\lim_{x\to\infty}\left(\frac{x^2}{x^2+x+1}\right)^{1+2x} = \lim_{x\to\infty}\left(1+\frac{x^2}{x^2+x+1}-1\right)^{1+2x} = \lim_{x\to\infty}\left(1+\frac{-x-1}{x^2+x+1}\right)^{1+2x}$$

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$$= \lim_{x\to\infty}\left(\underbrace{\left(1+\frac{-x-1}{x^2+x+1}\right)^{\frac{1}{\frac{-x-1}{x^2+x+1}}}}_{e}\right)^{\frac{-x-1}{x^2+x+1}\cdot(1+2x)} = e^{\lim_{x\to\infty}\frac{(-x-1)\cdot(1+2x)}{x^2+x+1}} = e^{\lim_{x\to\infty}\frac{-2x^2-3x-1}{x^2+x+1}} = e^{-2}$$

$$\boxed{\lim_{t\to 0}(1+t)^{\frac{1}{t}} = e}$$

2. $$\lim_{x\to 0}(\cos x)^{\frac{\operatorname{ctg} x}{x}} = \lim_{x\to 0}\left(1+\underbrace{\cos x-1}_{0}\right)^{\frac{\operatorname{ctg} x}{x}} = \lim_{x\to 0}\left(\underbrace{\left(1+\cos x-1\right)^{\frac{1}{\cos x-1}}}_{e}\right)^{(\cos x-1)\cdot\frac{\operatorname{ctg} x}{x}}$$

$$= e^{\lim_{x\to 0}(\cos x-1)\cdot\frac{\cos x}{x\sin x}} = e^{\lim_{x\to 0}\underbrace{\frac{\cos x-1}{x^2}}_{-\frac{1}{2}}\cdot\cancel{x^2}\cdot\frac{\overbrace{\cos x}^{=1 \text{ KAD } x\to 0}}{\cancel{x}\cdot\underbrace{\frac{\sin x}{x}}_{1}\cdot\cancel{x}}} = e^{-\frac{1}{2}} = \frac{1}{\sqrt{e}}$$

Za vežbu:

$$\lim_{x\to 0}(1+\operatorname{tg}x)^{\frac{1}{3x}} = e^{\frac{1}{3}}$$

$$\lim_{x\to 0}(x+e^{x})^{\frac{1}{\sin x}} = e^{2}$$

$$\lim_{x\to 0}(2-\cos x)^{\frac{3}{4x^2}} = e^{\frac{3}{8}}$$

$$\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{\frac{\sin x}{x-\sin x}} = e^{-1}$$

1.

$$\lim_{x\to 0}\frac{e^{\sqrt[3]{1+3x^2}}-e}{1-\cos x} = \lim_{x\to 0}\frac{e\left(e^{\sqrt[3]{1+3x^2}-1}-1\right)}{1-\cos x} =$$

$$\boxed{\lim_{t\to 0}\frac{e^{t}-1}{t}=1}$$

$$= \lim_{x\to 0}\frac{e\cdot\frac{e^{\sqrt[3]{1+3x^2}-1}-1}{\sqrt[3]{1+3x^2}-1}\cdot\left(\sqrt[3]{1+3x^2}-1\right)}{\underbrace{\frac{1-\cos x}{x^2}}_{\frac{1}{2}}\cdot x^2} = 2e\cdot\lim_{x\to 0}\frac{\sqrt[3]{1+3x^2}-1}{x^2}\cdot\frac{\sqrt[3]{(1+3x^2)^2}+\sqrt[3]{1+3x^2}+1}{\sqrt[3]{(1+3x^2)^2}+\sqrt[3]{1+3x^2}+1} =$$

$\to 1$

$$= 2e\cdot\lim_{x\to 0}\frac{\cancel{1}+3\cancel{x^2}-\cancel{1}}{\cancel{x^2}\cdot\underbrace{\left(\sqrt[3]{(1+3x^2)^2}+\sqrt[3]{1+3x^2}+1\right)}_{3}} = 2e\cdot\frac{3}{3} = 2e$$

2. $$\lim_{x\to 0}\frac{\ln(1+\sin^2 x)}{e^{x^2}-1}=\lim_{x\to 0}\frac{\overbrace{\frac{\ln(1+\sin^2 x)}{\sin^2 x}}^{\to 1}\cdot\overbrace{\sin^2 x}^{1}}{\underbrace{\frac{e^{x^2}-1}{x^2}}_{1}\cdot x^2}=1$$

$$\lim_{x\to 0}\frac{\ln(1+x)}{x}=1$$

3. $$\lim_{x\to 0^+}\frac{\ln(1+2x)}{\ln(1+x^2)}=\lim_{x\to 0^+}\frac{\overbrace{\frac{\ln(1+2x)}{2x}}^{1}\cdot 2x}{\underbrace{\frac{\ln(1+x^2)}{x^2}}_{1}\cdot x^2}=\lim_{x\to 0^+}\frac{2}{x}=+\infty$$

Primena limesa kod određivanja asimptota funkcije

Naći asimptote funkcija:

a) $f(x)=\sqrt{x^2+3x}$

* Kako je funkcija definisana za $x^2+3x\geq 0$, tj. $x\in(-\infty,-3]\cup[0,+\infty)$ sledi da je domen funkcije $D: x\in(-\infty,-3]\cup[0,+\infty)$ pa funkcija nema vertikalnu asimptotu.

* $\lim\limits_{x\to\infty} f(x)=+\infty$, $\lim\limits_{x\to-\infty} f(x)=+\infty$ $\Rightarrow$ funkcija nema horizontalnu asimptotu

* kosa asimptota $y=kx+n$

$k=\lim\limits_{x\to\pm\infty} f(x)$ $\qquad n=\lim\limits_{x\to\pm\infty}(f(x)-kx)$

$$\lim_{x\to\pm\infty}\frac{f(x)}{x}=\lim_{x\to\pm\infty}\frac{\sqrt{x^2+3x}}{x}=\lim_{x\to\pm\infty}\frac{|x|\cdot\sqrt{1+\frac{3}{x}}}{x}=\begin{cases}\lim\limits_{x\to+\infty}\frac{x\cdot\sqrt{1+\frac{3}{x}}}{x}=1 \Rightarrow k_1=1\\[2ex] \lim\limits_{x\to-\infty}\frac{-x\cdot\sqrt{1+\frac{3}{x}}}{x}=-1 \Rightarrow k_2=-1\end{cases}$$

$$n_1=\lim_{x\to\infty}(f(x)-k_1x)=\lim_{x\to\infty}\left(\sqrt{x^2+3x}-x\right)\cdot\frac{\sqrt{x^2+3x}+x}{\sqrt{x^2+3x}+x}$$

$$=\lim_{x\to\infty}\frac{\cancel{x^2}+3x-\cancel{x^2}}{\sqrt{x^2+3x}+x}=\lim_{x\to\infty}\frac{3x}{\sqrt{x^2+3x}+x}=\frac{3}{2}\Rightarrow \boxed{y=x+\frac{3}{2}\text{ je desna kosa asimptota}}$$

$$n_2=\lim_{x\to-\infty}(f(x)-k_2x)=\lim_{x\to-\infty}\left(\sqrt{x^2+3x}+x\right)\cdot\frac{\sqrt{x^2+3x}-x}{\sqrt{x^2+3x}-x}=\lim_{x\to-\infty}\frac{\cancel{x^2}+3x-\cancel{x^2}}{\sqrt{x^2+3x}-x}=\lim_{x\to-\infty}\frac{3x}{\sqrt{x^2+3x}-x}=$$

$$=-\frac{3}{2}$$

$y=-x-\frac{3}{2}$ je leva kosa asimptota

2. $f(x) = \operatorname{arctg}\left(\frac{x-1}{\sqrt{x^2-9}}\right)$

Domen: $x \in (-\infty, -3) \cup (3, +\infty)$

$$\lim_{x\to -3} \operatorname{arctg}\left(\frac{x-1}{\sqrt{x^2-9}}\right) = \operatorname{arctg}(-\infty) = -\frac{\pi}{2}$$

$$\lim_{x\to 3^+} \operatorname{arctg}\left(\frac{x-1}{\sqrt{x^2-9}}\right) = \operatorname{arctg}(+\infty) = \frac{\pi}{2}$$

$\Rightarrow$ funkcija nema vertikalnu asimptotu

$$\lim_{x\to\infty} \operatorname{arctg}\left(\frac{x-1}{\sqrt{x^2-9}}\right) = \operatorname{arctg} 1 = \frac{\pi}{4} \Rightarrow y = \frac{\pi}{4} \text{ horizontalna asimptota}$$

$$\lim_{x\to -\infty} \operatorname{arctg} \frac{x-1}{\sqrt{x^2-9}} = \operatorname{arctg}(-1) = -\frac{\pi}{4} \Rightarrow y = -\frac{\pi}{4} \text{ horizontalna asimptota}$$

Funkcija nema kosu asimptotu

Zadaci za vežbu (naći asimptote)

1. $f(x) = \sqrt{4-x^2}$
2. $f(x) = 3x + \sqrt{x^2+4x}$
3. $f(x) = \sqrt[3]{x^3-2x^2+1}$

$$f(x) = \frac{e^{2010x}-2}{e^{2011x}+1}$$

Domen: $x \in \mathbb{R} \Rightarrow$ nema vertikalnu

$$\lim_{x\to\infty} \frac{e^{2010x}-2}{e^{2011x}+1} = \lim_{x\to\infty} \frac{e^{2010x}\left(1-\frac{2}{e^{2010x}}\right)}{e^{2011x}\left(1+\frac{1}{e^{2011x}}\right)} =$$

$$= \lim_{x\to+\infty} \frac{1-\cancel{\frac{2}{e^{2010x}}}}{e^{x}\left(1+\cancel{\frac{1}{e^{2011x}}}\right)} = 0 \Rightarrow y = 0 \text{ horiz. asimpt.}$$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{e^{2010x}-2}{e^{2011x}+1} = -2 \Rightarrow y = -2 \text{ horiz. asimpt.}$$