# НЕСВОЈСТВЕНИ ИНТЕГРАЛИ

- Одређени интеграл: $\int_a^b f(x)\,dx$

1) $-\infty < a, b < \infty$

2) $f : [a,b] \to \mathbb{R}$ ограничена

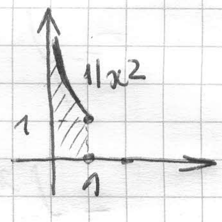
Ако неки од ових услова није испуњен, онда проширујемо појам интеграла, уводимо несвојствени интеграл.

- 1) $\int_{-\infty}^{b} f(x)\,dx$; $\int_{a}^{+\infty} f(x)\,dx$; $\int_{-\infty}^{+\infty} f(x)\,dx$

→ несвојствени интеграли прве врсте

- 2) $f$ није ограничена, нпр. $\int_{\boxed{0}}^{1} \frac{dx}{x^2}$

→ несвојствени интеграли друге врсте




$x = 0 \Rightarrow \frac{1}{x^2} = \infty$

0 – сингуларитет

- Ако је решење реалан број, онда кажемо да интеграл конвергира. (К)

Ако је решење $+\infty$ или $-\infty$, онда кажемо да интеграл одређено дивергира. (ОД)

Ако нема решења, онда кажемо да интеграл дивергира. (Д)

- Особине. $I_1 = \int_a^b f(x)\,dx$, $I_2 = \int_a^b g(x)\,dx$, $b$ – сингуларитет

⊛ $I_1$ (К): $\int_a^b k f(x)\,dx = k \int_a^b f(x)\,dx$, $k \in \mathbb{R}$

⊛ $I_1, I_2$ (К): $\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$.

ЕКВИКОНВЕРГЕНТНИ

(*) $a<c<b: \quad \int_a^b f(x)dx$ к $\Longleftrightarrow \int_c^b f(x)dx$ к

и важи: $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$

• За два интеграла кажемо да су еквиконвергентни ако ~~[illegible]~~ истовремено оба конвергирају или истовремено оба дивергирају.

• к + к = к; к + д = д; д + д = ? (не знамо)

1. а) $I=\int_{-\infty}^{0} e^x dx = \lim_{\alpha\to-\infty} \underbrace{\int_\alpha^0 e^x dx} = \lim_{\alpha\to-\infty} [e^x]\Big|_\alpha^0 = \lim_{\alpha\to-\infty}(1-e^\alpha)$

$= 1-\lim_{\alpha\to-\infty} e^\alpha = 1-0 = 1$ к

б) $I=\int_0^{\infty} e^{-x}dx = \lim_{\beta\to\infty}\underbrace{\int_0^\beta e^{-x}dx} = \lim_{\beta\to\infty}[-e^{-x}]\Big|_0^\beta =$

$= \lim_{\beta\to\infty}(-e^{-\beta}+1) = -\lim_{\beta\to\infty} e^{-\beta}+1 = 0+1 = 1$ к

в) $I=\int_{-\infty}^{\infty} e^{-|x|}dx = \int_{-\infty}^{0} e^{-|x|}dx + \int_0^\infty e^{-|x|}dx =$

$= \int_{-\infty}^0 e^x dx + \int_0^\infty e^{-x}dx \overset{а), б)}{=} 1+1 = 2.$ к

2. $I=\int_{\boxed{-1}}^{\boxed{1}} \frac{dx}{\sqrt{1-x^2}} = \int_{\boxed{-1}}^{0}\frac{dx}{\sqrt{1-x^2}} + \int_0^{\boxed{1}}\frac{dx}{\sqrt{1-x^2}} =$

$= \lim_{\alpha\to-1+}\int_\alpha^0 \frac{dx}{\sqrt{1-x^2}} + \lim_{\beta\to1-}\int_0^\beta\frac{dx}{\sqrt{1-x^2}} = \lim_{\alpha\to-1+}[\arcsin x]\Big|_\alpha^0 + \lim_{\beta\to1-}[\arcsin x]\Big|_0^\beta$

$= \lim_{\alpha\to-1+}(0-\arcsin\alpha) + \lim_{\beta\to1-}(\arcsin\beta - 0) = -\left(-\frac{\pi}{2}\right)+\frac{\pi}{2} = \pi.$ к

3. Испитати конвергенцију несвојствених интеграла.

а) $I = \int_{\boxed{0}}^{1} \frac{dx}{x^k}, \quad k > 0$

<u>$k=1$:</u> $I = \int_{\boxed{0}}^{1} \frac{dx}{x^k} = \lim_{\varepsilon \to 0+} \int_{\varepsilon}^{1} \frac{dx}{x} = \lim_{\varepsilon \to 0+} \left[\ln|x|\right]\Big|_{\varepsilon}^{1} =$

$= \lim_{\varepsilon \to 0+} (0 - \ln \varepsilon) = +\infty \quad \text{ОД}$

<u>$k \neq 1$:</u> $I = \int_{\boxed{0}}^{1} \frac{dx}{x^k} = \lim_{\varepsilon \to 0+} \int_{\varepsilon}^{1} \frac{dx}{x^k} = \lim_{\varepsilon \to 0+} \left[\frac{x^{1-k}}{1-k}\right]\Big|_{\varepsilon}^{1} =$

$= \frac{1}{1-k} \lim_{\varepsilon \to 0+} \left(1 - \varepsilon^{1-k}\right) = \frac{1}{1-k}\left(1 - \lim_{\varepsilon \to 0+} \varepsilon^{1-k}\right) =$

$= \begin{cases} \frac{1}{1-k}, & k < 1 \quad \text{К} \\ +\infty, & k > 1 \quad \text{ОД} \end{cases}$

$$\underset{(k>0)}{I} = \int_0^1 \frac{dx}{x^k} = \begin{cases} \frac{1}{1-k}, & k < 1 \quad \text{К} \\ +\infty, & k \geq 1 \quad \text{ОД} \end{cases}$$

а1) $I = \int_{\boxed{0}}^{1} \frac{dx}{x^{1/2}} = \lim_{\varepsilon \to 0+} \underline{\int_{\varepsilon}^{1} x^{-1/2}\, dx} = \lim_{\varepsilon \to 0+} \frac{x^{1/2}}{1/2}\Big|_{\varepsilon}^{1} =$

$= \lim_{\varepsilon \to 0+} 2\left(1 - \varepsilon^{1/2}\right) = 2 \quad \text{К}$

а2) $I = \int_{\boxed{0}}^{1} \frac{dx}{x^3} = \lim_{\varepsilon \to 0+} \int_{\varepsilon}^{1} x^{-3}\, dx = \lim_{\varepsilon \to 0+} \left[\frac{x^{-2}}{-2}\right]\Big|_{\varepsilon}^{1} =$

$= -\frac{1}{2} \lim_{\varepsilon \to 0+} \left(1 - \frac{1}{\varepsilon^2}\right) = +\infty \quad \text{ОД}$

б) $I = \int_1^{\infty} \frac{dx}{x^k}, \quad k>0$

$\underline{k=1}$: $I = \int_1^{\infty} \frac{dx}{x} = \lim\limits_{\beta\to\infty} \int_1^{\beta} \frac{dx}{x} = \lim\limits_{\beta\to\infty} \left[\ln|x|\right]\Big|_1^{\beta} =$

$= \lim\limits_{\beta\to\infty} (\ln\beta - 0) = \infty$ OB

$\underline{k\neq 1}$: $I = \int_1^{\infty} \frac{dx}{x^k} = \lim\limits_{\beta\to\infty} \int_1^{\beta} x^{-k}\,dx = \lim\limits_{\beta\to\infty} \left[\frac{x^{1-k}}{1-k}\right]\Big|_1^{\beta} =$

$= \frac{1}{1-k} \lim\limits_{\beta\to\infty} \left(\beta^{1-k} - 1\right) = \frac{1}{1-k} \left(\lim\limits_{\beta\to\infty} \beta^{1-k} - 1\right) =$

$$= \begin{cases} \frac{1}{k-1}, & k>1 \quad \text{K} \\ +\infty, & k<1 \quad \text{OB} \end{cases}$$

$$I = \int_1^{\infty} \frac{dx}{x^k} = \begin{cases} \frac{1}{k-1}, & k>1 \quad \text{K} \\ +\infty, & k\le 1 \quad \text{OB} \end{cases}$$

б$_1$) $I = \int_1^{\infty} \frac{dx}{x^{1/3}} = \lim\limits_{\beta\to\infty} \int_1^{\beta} x^{-1/3}\,dx = \lim\limits_{\beta\to\infty} \left[\frac{x^{2/3}}{2/3}\right]\Big|_1^{\beta} =$

$= \frac{3}{2} \lim\limits_{\beta\to\infty} \left(\beta^{2/3} - 1\right) = +\infty$ OB

б$_2$) $I = \int_1^{\infty} \frac{dx}{x^2} = \lim\limits_{\beta\to\infty} \int_1^{\beta} x^{-2}\,dx = \lim\limits_{\beta\to\infty} \left[\frac{x^{-1}}{-1}\right]\Big|_1^{\beta} =$

$= -\lim\limits_{\beta\to\infty} \left(\beta^{-1} - 1\right) = 1.$ K

в) $I = \int_{\boxed{0}}^{\infty} \frac{dx}{x^k}, \quad k>0$

$$I = \int_0^{\infty} \frac{dx}{x^k} = \underbrace{\int_0^1 \frac{dx}{x^k}}_{I_1} + \underbrace{\int_1^{\infty} \frac{dx}{x^k}}_{I_2}$$

$\underline{k<1}$: $I_1$ K $+ I_2$ OB $= I$ OB

$\underline{k=1}$: $I_1$ OB $+ I_2$ OB $= \infty + \infty = \infty = I$ OB

$\underline{k>1}$: $I_1$ OB $+ I_2$ K $= I$ OB

$I = +\infty, \ \forall k>0$; OB

4. а) $I_n = \int_0^{\infty} x^n e^{-x} dx, \quad n \in \mathbb{N}_0$ – рекурентна формула?

б) $I_5 = \int_0^{\infty} x^5 e^{-x} dx = ?$

$$I_n = \int_0^{\infty} x^n e^{-x} dx = \lim_{\beta \to \infty} \int_0^{\beta} x^n e^{-x} dx = \left\{ \begin{matrix} u = x^n, & dv = e^{-x} dx \\ du = n x^{n-1} dx, & v = -e^{-x} \end{matrix} \right\} =$$

$$= \lim_{\beta \to \infty} \left( \left[ -x^n e^{-x} \right] \Big|_0^{\beta} + n \int_0^{\beta} x^{n-1} e^{-x} dx \right) =$$

$$= \lim_{\beta \to \infty} \left( -\beta^n e^{-\beta} + 0 \cdot 1 \right) + n \lim_{\beta \to \infty} \int_0^{\beta} x^{n-1} e^{-x} dx =$$

$$= - \lim_{\beta \to \infty} \frac{\beta^n}{e^{\beta}} + n \underbrace{\int_0^{\infty} x^{n-1} e^{-x} dx}_{I_{n-1}},$$

$$\lim_{\beta \to \infty} \frac{\beta^n}{e^{\beta}} = \left\{ \frac{\infty}{\infty} \right\} \overset{L}{=} \lim_{\beta \to \infty} \frac{n \beta^{n-1}}{e^{\beta}} = \left\{ \frac{\infty}{\infty} \right\} \overset{L}{=} \lim_{\beta \to \infty} \frac{n(n-1) \beta^{n-2}}{e^{\beta}} = \left\{ \frac{\infty}{\infty} \right\} \overset{L}{=} \cdots$$

$$\cdots \overset{L}{=} \lim_{\beta \to \infty} \frac{n(n-1) \cdots 2 \cdot 1 \, \overset{=1}{\beta^0}}{e^{\beta}} = \lim_{\beta \to \infty} \frac{n!}{e^{\beta}} = 0$$

$$I_n = n I_{n-1} = n(n-1) I_{n-2} = n(n-1)(n-2) I_{n-3} = \cdots$$

$$\cdots = n(n-1)(n-2) \cdots 2 \cdot 1 \cdot I_0 = n! \cdot I_0$$

$$I_0 = \int_0^{\infty} x^0 e^{-x} dx = \int_0^{\infty} e^{-x} dx \overset{1.\delta}{=} 1$$

$$\boxed{I_n = n!}$$

$$I_5 = 5! = 120$$

5. a) $I_n = \int_0^\infty x^{2n+1} e^{-x^2}\,dx$, $n \in \mathbb{N}_0$ – rekurentna formula?

b) $I_6 = \int_0^\infty x^{13} e^{-x^2}\,dx = ?$

<u>1. način:</u>

$$I_n = \int_0^{\infty} x^{2n+1} e^{-x^2}\,dx = \lim_{\beta\to\infty}\int_0^\beta (x^2)^n e^{-x^2}\, x\,dx =$$

$$= \left\{\begin{array}{l} x^2 = t \\ 2x\,dx = dt \end{array}\right\} = \lim_{\beta\to\infty}\int_0^{\beta^2} t^n e^{-t}\,\frac{dt}{2} = \cancel{\boxed{4}}$$

$$= \frac{1}{2}\int_0^\infty t^n e^{-t}\,dt = \boxed{4} = \frac{1}{2}\,n!$$

<u>2. način:</u>

$$I_n = \int_0^{\infty} x^{2n+1} e^{-x^2}\,dx = \lim_{\beta\to\infty}\int_0^\beta x^{2n} e^{-x^2}\, x\,dx =$$

$$= \left\{\begin{array}{l} u = x^{2n},\ dv = e^{-x^2}x\,dx \\ du = 2n\,x^{2n-1} \\ v = \int x e^{-x^2}dx = \left\{\begin{array}{l} -x^2 = t \\ -2x\,dx = dt \end{array}\right\} = -\frac{1}{2}\int e^t dt = -\frac{1}{2}e^t = -\frac{1}{2}e^{-x^2} \end{array}\right\} =$$

$$= \lim_{\beta\to\infty}\left(\left[-\frac{1}{2}x^{2n}e^{-x^2}\right]\Big|_0^\beta + n\int_0^\beta x^{2n-1}e^{-x^2}dx\right) =$$

$$= -\frac{1}{2}\lim_{\beta\to\infty}\left(\beta^{2n}e^{-\beta^2} - 0\cdot 1\right) + n\lim_{\beta\to\infty}\int_0^\beta x^{2n-1}e^{-x^2}dx =$$

$$= -\frac{1}{2}\lim_{\beta\to\infty}\frac{\beta^{2n}}{e^{\beta^2}} + n\,\cancel{\lim}\underbrace{\int_0^\infty x^{2(n-1)+1}e^{-x^2}dx}_{I_{n-1}}$$

$$\lim_{\beta\to\infty}\frac{\beta^{2n}}{e^{\beta^2}} = \left\{\frac{\infty}{\infty},\ \text{Лопитал}\right\} = \lim_{\beta\to\infty}\frac{\cancel{2}n\,\beta^{2n-1}}{\cancel{2}\beta\, e^{\beta^2}} = \left\{\frac{\infty}{\infty},\ \text{Лопитал}\right\} =$$

$$= \lim_{\beta\to\infty}\frac{2n(2n-1)\beta^{2n-2}}{2(e^{\beta^2}+\beta\cdot 2\beta e^{\beta^2})} = \left\{\frac{\infty}{\infty},\ \text{Лопитал}\right\} = \dots$$

$$\dots = \left\{\frac{\infty}{\infty},\ \text{Лопитал}\right\} = \lim_{\beta\to\infty}\frac{2n(2n-1)\dots 2\cdot 1\cdot \overbrace{\beta^0}^{=1}}{\underbrace{f(\beta)}_{=P_{2n}(\beta)e^{\beta^2}\to\infty,\ \beta\to\infty}} = \lim_{\beta\to\infty}\frac{(2n)!}{f(\beta)} = 0$$

$$I_n = n I_{n-1} = n(n-1) I_{n-2} = \dots = n(n-1)\dots\cdot 2\cdot 1\cdot I_0 = n!\, I_0$$

$$I_0 = \int_0^{\infty} x e^{-x^2}dx = \lim_{\beta\to\infty}\int_0^{\beta} x e^{-x^2}dx = \underbrace{\dots}_{\text{видети } \sigma = \int x e^{-x^2}dx} = \lim_{\beta\to\infty}\left[-\frac{1}{2}e^{-x^2}\right]_0^{\beta} =$$

$$= -\frac{1}{2}\lim_{\beta\to\infty}(\overbrace{e^{-\beta^2}}^{0} - 1) = \frac{1}{2}$$

$$\boxed{I_n = \frac{1}{2}n!}$$

$$I_6 = \frac{1}{2}6! = 360$$