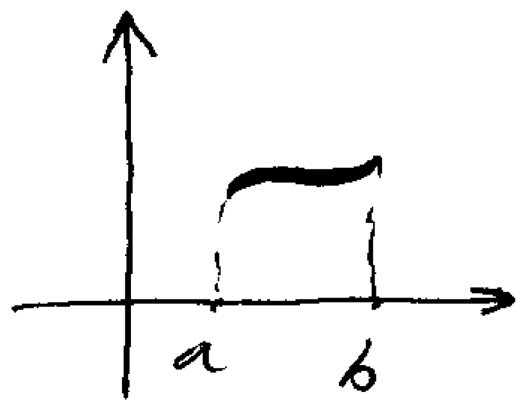
III Дужина лука криве



1) **Декартове координате:** $y=f(x),\ x\Big|_a^b: \quad L=\int_a^b\sqrt{1+(f'(x))^2}\,dx$

$x=g(y),\ y\Big|_c^d: \quad L=\int_c^d\sqrt{1+(g'(y))^2}\,dy$

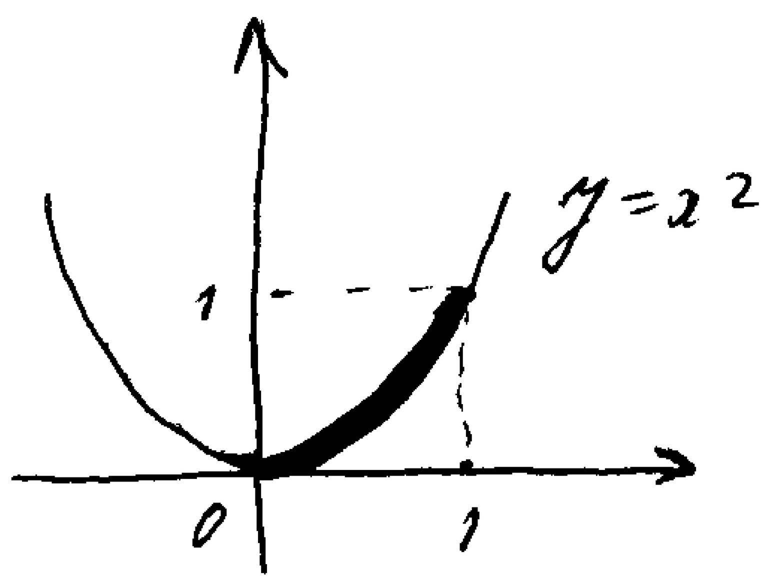
2) **Крива задата параметарски:** $x=x(t),\ y=y(t),\ t\Big|_\alpha^\beta$

$$L=\int_\alpha^\beta\sqrt{(x'(t))^2+(y'(t))^2}\,dt$$

3) **Поларне координате:** $x=\rho\cos\varphi,\ y=\rho\sin\varphi,\ \rho=\rho(\varphi),\ \varphi\Big|_\alpha^\beta$

$$L=\int_\alpha^\beta\sqrt{(\rho'(\varphi))^2+(\rho(\varphi))^2}\,d\varphi$$

* Израчунати дужину лука криве $y=x^2$ од тачке $(0,0)$ до тачке $(1,1)$.



$$L=\int_0^1\sqrt{1+((x^2)')^2}\,dx=\int_0^1\sqrt{1+(2x)^2}\,dx=\left\{\begin{array}{l}2x=\operatorname{sh}t\\ 2dx=\operatorname{ch}t\,dt\end{array}\right\}$$

$$=\frac12\int_0^{\operatorname{arsh}2}\underbrace{\sqrt{1+\operatorname{sh}^2t}}_{\operatorname{ch}^2t}\cdot\operatorname{ch}t\,dt=\frac12\int_0^{\ln(2+\sqrt5)}\operatorname{ch}^2t\,dt=$$

$$=\frac12\int_0^{\ln(2+\sqrt5)}\left(\frac{e^t+e^{-t}}{2}\right)^2dt=\frac18\int_0^{\ln(2+\sqrt5)}\left(e^{2t}+2+e^{-2t}\right)dt=$$

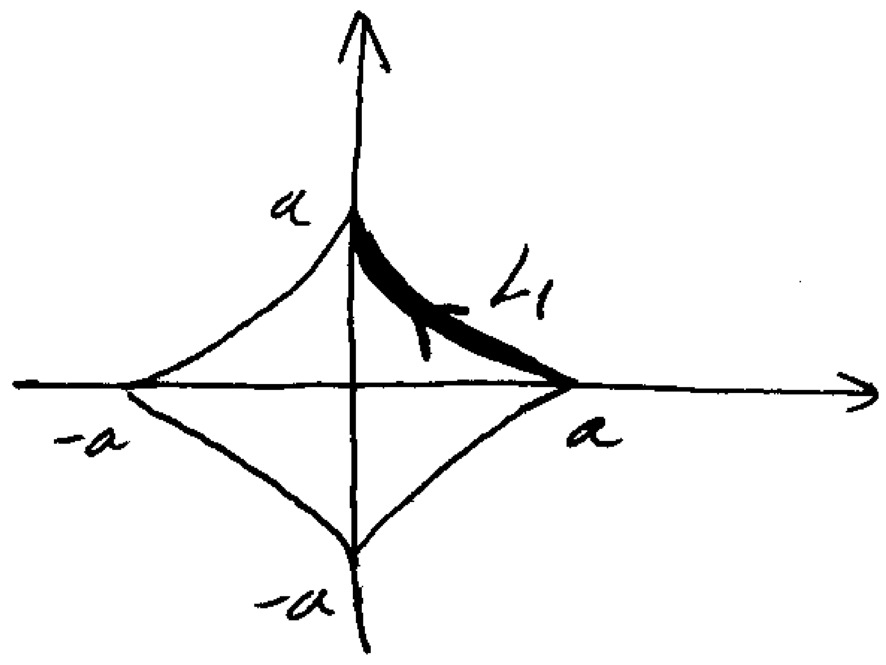
$$=\frac18\left[\frac12e^{2t}+2t-\frac12e^{-2t}\right]\Bigg|_0^{\ln(2+\sqrt5)}=$$

$$=\frac18\left(\frac12e^{2\ln(2+\sqrt5)}+2\ln(2+\sqrt5)-\frac12e^{-2\ln(2+\sqrt5)}\cancel{-\frac12}-0\cancel{+\frac12}\right)$$

$$=\frac1{16}(2+\sqrt5)^2+\frac14\ln(2+\sqrt5)-\frac1{16}\,\frac{1}{(2+\sqrt5)^2}=\frac1{16}\left(9+4\sqrt5-\frac{1}{9+4\sqrt5}\right)+\frac14\ln(2+\sqrt5)$$

$$=\left\{\frac{1}{9+4\sqrt5}\cdot\frac{9-4\sqrt5}{9-4\sqrt5}=9-4\sqrt5\right\}=\frac1{16}\cdot8\sqrt5+\frac14\ln(2+\sqrt5)=\frac14\left(2\sqrt5+\ln(2+\sqrt5)\right)$$

Израчунати дужину лука криве $\underbrace{x^{2/3}+y^{2/3}=a^{2/3}}_{\text{АСТРОИДА}}$ $(a>0)$.



$L = 4L_1$

параметризација:

$$x^{2/3}+y^{2/3}=a^{2/3} \quad /: a^{2/3}$$

$$\left(\left(\frac{x}{a}\right)^{1/3}\right)^2+\left(\left(\frac{y}{a}\right)^{1/3}\right)^2=1$$

$$\left(\frac{x}{a}\right)^{1/3}=\cos t, \quad \left(\frac{x}{a}\right)^{1/3}=\sin t$$

$$x=a\cos^3 t, \quad y=a\sin^3 t$$

1. квадрант: $t\Big|_0^{\pi/2}$

$$(x')^2+(y')^2=(-3a\cos^2 t\sin t)^2+(3a\sin^2 t\cos t)^2$$

$$=9a^2\cos^4 t\sin^2 t+9a^2\sin^4 t\cos^2 t$$

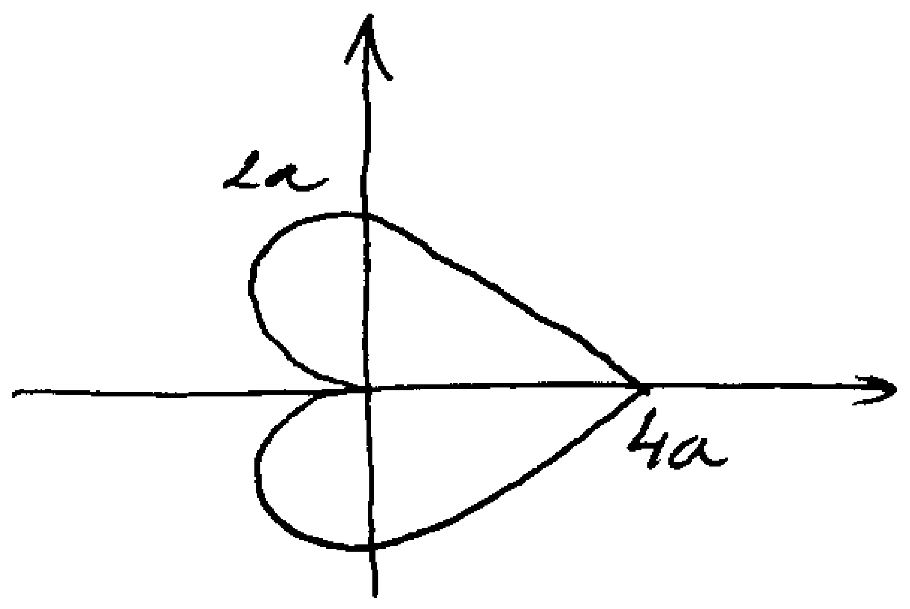
$$=9a^2\sin^2 t\cos^2 t\underbrace{(\cos^2 t+\sin^2 t)}_{=1}$$

$$L=4L_1=4\int_0^{\pi/2}\sqrt{9a^2\sin^2 t\cos^2 t}\,dt=4\cdot 3a\int_0^{\pi/2}|\sin t|\cdot|\cos t|\,dt=$$

$$=\left\{\begin{array}{l} t\big|_0^{\pi/2},\ \sin t,\cos t\geq 0 \\ |\sin t|=\sin t,\ |\cos t|=\cos t\end{array}\right\}=12a\int_0^{\pi/2}\sin t\cos t\,dt=6a\int_0^{\pi/2}\sin 2t\,dt=$$

$$=6a\left[-\frac{1}{2}\cos 2t\right]\Big|_0^{\pi/2}=-3a(-1-1)=6a$$

⊛ Израчунати дужину лука криве $\underbrace{\rho = 2a(1+\cos\varphi)}_{\text{КАРДИОИДА}}$ $(a>0)$



$$\varphi\Big|_0^{2\pi}$$

$$\rho' = -2a\sin\varphi$$

$$\rho'^2+\rho^2 = 4a^2\sin^2\varphi + 4a^2(1+2\cos\varphi+\cos^2\varphi)$$

$$= 4a^2(\sin^2\varphi + 1 + 2\cos\varphi + \cos^2\varphi)$$

$$= 8a^2(1+\cos\varphi)$$

$$L = \int_0^{2\pi}\sqrt{8a^2(1+\cos\varphi)}\,d\varphi = 2a\sqrt{2}\int_0^{2\pi}\sqrt{1+\cos\varphi}\,d\varphi = 2a\sqrt{2}\int_0^{2\pi}\sqrt{2\cos^2\frac{\varphi}{2}}\,d\varphi =$$

$$= 4a\int_0^{2\pi}\left|\cos\frac{\varphi}{2}\right|d\varphi = \left\{\begin{array}{l} \varphi\Big|_0^{2\pi},\ \frac{\varphi}{2}\Big|_0^{\pi} \\ \varphi\Big|_0^{\pi} \to \frac{\varphi}{2}\Big|_0^{\pi/2},\ \cos\frac{\varphi}{2}\geq 0,\ \left|\cos\frac{\varphi}{2}\right| = \cos\frac{\varphi}{2} \\ \varphi\Big|_\pi^{2\pi} \to \frac{\varphi}{2}\Big|_{\pi/2}^{\pi},\ \cos\frac{\varphi}{2} < 0,\ \left|\cos\frac{\varphi}{2}\right| = -\cos\frac{\varphi}{2} \end{array}\right\} =$$

$$= 4a\left(\int_0^{\pi}\cos\frac{\varphi}{2}\,d\varphi + \int_\pi^{2\pi}\left(-\cos\frac{\varphi}{2}\right)d\varphi\right) =$$

$$= 4a\left(\left[2\sin\frac{\varphi}{2}\right]\Big|_0^{\pi} - \left[2\sin\frac{\varphi}{2}\right]\Big|_\pi^{2\pi}\right) = 4a(2+2) = 16a$$