IV Површина обртних површи

1) Декартове координате:

$$P_x = 2\pi \int_a^b |f(x)| \sqrt{1+(f'(x))^2}\, dx$$

$$P_y = 2\pi \int_c^d |g(y)| \sqrt{1+(g'(y))^2}\, dy$$

2) Крива задата параметарски: $x = x(t)$, $y = y(t)$, $t \Big|_\alpha^\beta$

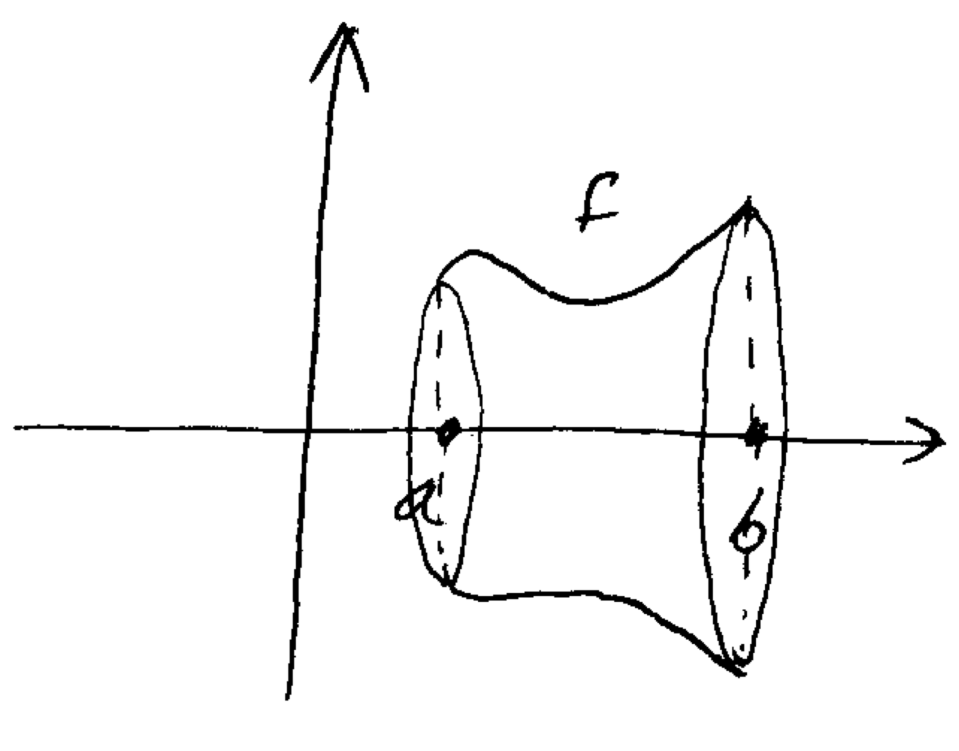
$$P_x = 2\pi \int_\alpha^\beta |y(t)| \sqrt{(x'(t))^2+(y'(t))^2}\, dt$$

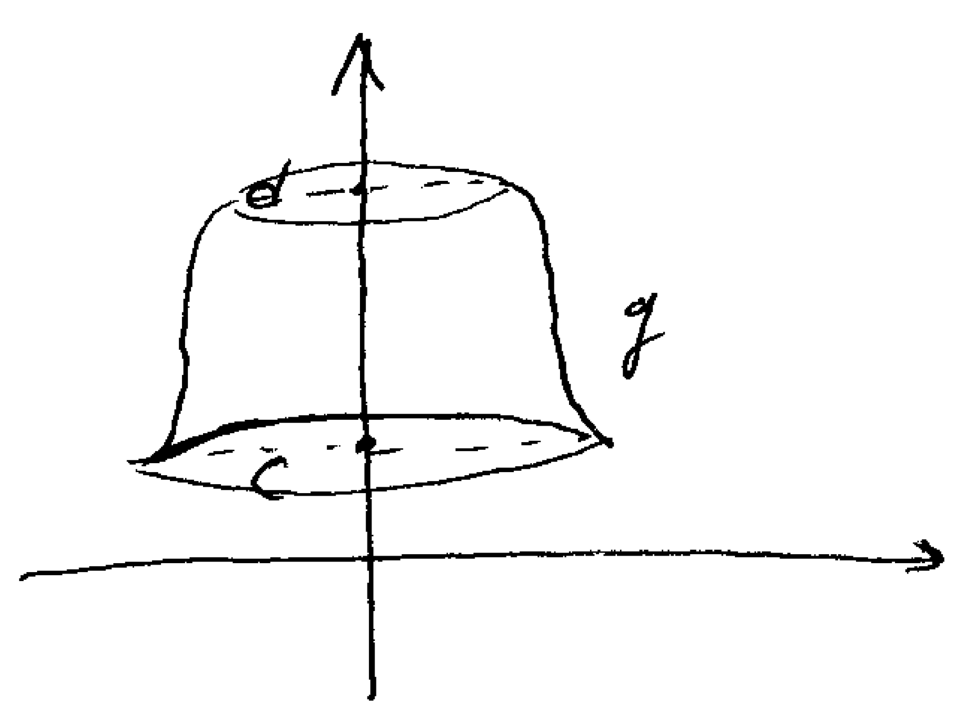
$$P_y = 2\pi \int_\alpha^\beta |x(t)| \sqrt{(x'(t))^2+(y'(t))^2}\, dt$$

3) Поларне координате: $x = \rho\cos\varphi$, $y = \rho\sin\varphi$, $\rho = \rho(\varphi)$, $\varphi \Big|_\alpha^\beta$

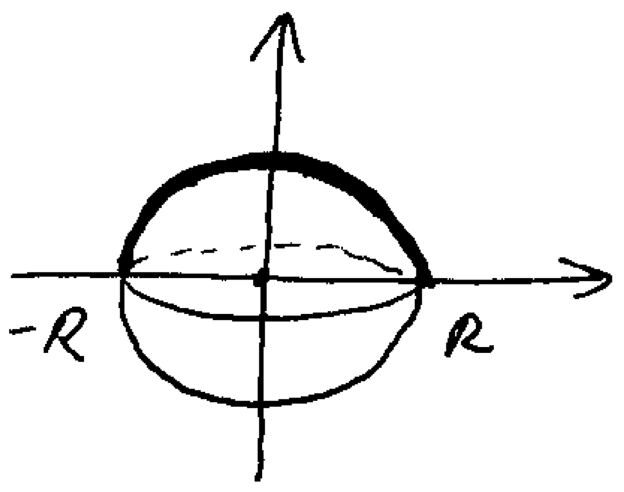
$$P_x = 2\pi \int_\alpha^\beta \rho\, |\sin\varphi| \sqrt{(\rho'(\varphi))^2+(\rho(\varphi))^2}\, d\varphi$$

$$P_y = 2\pi \int_\alpha^\beta \rho\, |\cos\varphi| \sqrt{(\rho'(\varphi))^2+(\rho(\varphi))^2}\, d\varphi$$





(*) Применом одређеног интеграла израчунати површину лопте полупречника R.



$x \Big|_{-R}^{R}$; $x^2+y^2=R^2$, $y=\oplus\sqrt{R^2-x^2}$

(Узимамо само горњу полукружницу, јер кад бисмо узели целу кружницу, добили бисмо дупло већу површину)

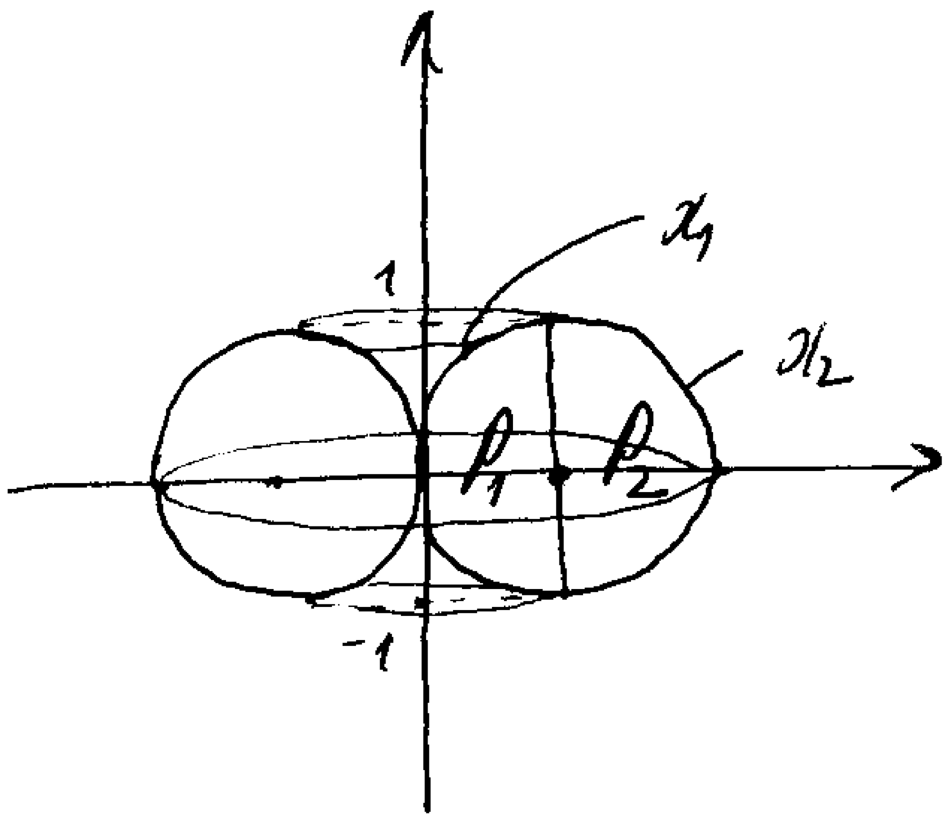
$P = 4R^2\pi$

$$y' = \frac{-\cancel{2}x}{\cancel{2}\sqrt{R^2-x^2}} \; ; \; (y')^2 = \frac{x^2}{R^2-x^2} \; ; \; 1+(y')^2 = \frac{R^2\cancel{-x^2}\cancel{+x^2}}{R^2-x^2}$$

$$P_x = 2\pi\int_{-R}^{R} \left|\sqrt{R^2-x^2}\right| \cdot \sqrt{\frac{R^2}{R^2-x^2}}\,dx = 2\pi\cdot 2\int_0^R \cancel{\sqrt{R^2-x^2}}\cdot\frac{R}{\cancel{\sqrt{R^2-x^2}}}\,dx =$$

$$= 4R\pi \underbrace{\int_0^R dx}_{=R} = 4R^2\pi$$

(*) Израчунати површину тела које настаје ротацијом око y-осе криве $(x-1)^2+y^2=1$.



$$(x-1)^2+y^2=1$$
$$(x-1)^2=1-y^2$$
$$x-1=\pm\sqrt{1-y^2}$$
$$x=1\pm\sqrt{1-y^2}$$

$y\,\Big|_{-1}^{1}$, горња: $x_2=1+\sqrt{1-y^2}$

доња: $x_1=1-\sqrt{1-y^2}$

$$P=P_1+P_2=2\pi\int_{-1}^{1}|x_1(y)|\sqrt{1+(x_1'(y))^2}\,dy+2\pi\int_{-1}^{1}|x_2(y)|\sqrt{1+(x_2'(y))^2}\,dy$$

$$x_1'=\frac{-2y}{2\sqrt{1-y^2}}=-\frac{y}{\sqrt{1-y^2}}\ ;\ (x_1')^2=\frac{y^2}{1-y^2}$$

$$x_2'=\frac{2y}{2\sqrt{1-y^2}}=\frac{y}{\sqrt{1-y^2}}\ ,\ (x_2')^2=\frac{y^2}{1-y^2}$$

$$\sqrt{1+(x_{1/2}'(y))^2}=\sqrt{1+\frac{y^2}{1-y^2}}=\sqrt{\frac{1\cancel{-y^2}\cancel{+y^2}}{1-y^2}}=\frac{1}{\sqrt{1-y^2}}$$

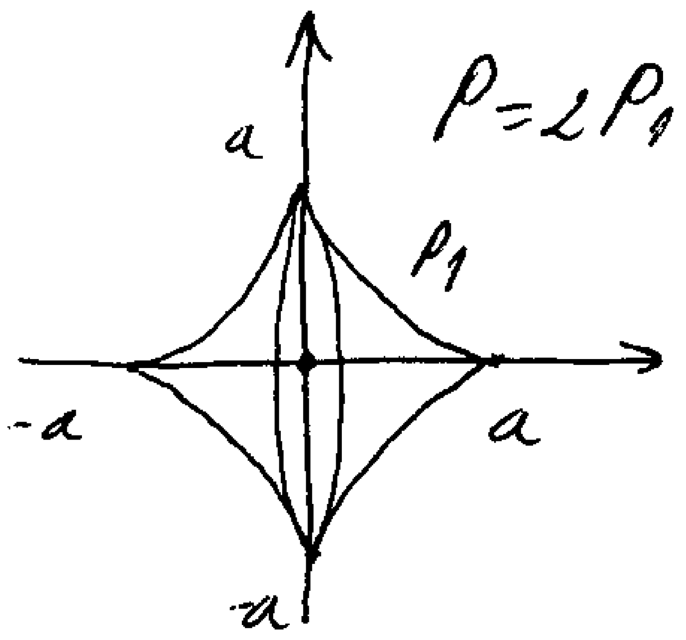
$$P=2\pi\int_{-1}^{1}\frac{1}{\sqrt{1-y^2}}\left(|1+\sqrt{1-y^2}|+|1-\sqrt{1-y^2}|\right)dy$$

$$=2\pi\cdot2\int_0^1\frac{1}{\sqrt{1-y^2}}\left(|1+\sqrt{1-y^2}|+|1-\sqrt{1-y^2}|\right)=\left\{\begin{matrix} y|_0^1, & y^2|_0^1, & 1-y^2|_0^1, \\ \sqrt{1-y^2}\,|_0^1, & 1+\sqrt{1-y^2}\,|_1^2 & \\ & 1-\sqrt{1-y^2}\,|_0^1 & \end{matrix}\right\}$$

$$=4\pi\int_0^1\frac{1}{\sqrt{1-y^2}}\left(1+\cancel{\sqrt{1-y^2}}+1-\cancel{\sqrt{1-y^2}}\right)dy=8\pi\int_0^1\frac{dy}{\sqrt{1-y^2}}=$$

$$=8\pi\,[\arcsin y]\,\Big|_0^1=8\pi\cdot\frac{\pi}{2}=4\pi^2$$

(*) Израчунати запремину тела које настаје ротацијом око x-осе криве $\underbrace{x^{2/3}+y^{2/3}=a^{2/3}}_{\text{АСТРОИДА}}$ $(a>0)$



$P=2P_1$

параметризујући: $x=a\cos^3 t$, $y=a\sin^3 t$

$t\Big|_0^{\pi/2}$ (1. квадрант)

$x'=-3a\cos^2 t\sin t$, $y'=3a\sin^2 t\cos t$

$$(x')^2+(y')^2=9a^2\cos^4 t\sin^2 t+9a^2\sin^4 t\cos^2 t$$

$$=9a^2\sin^2 t\cos^2 t$$

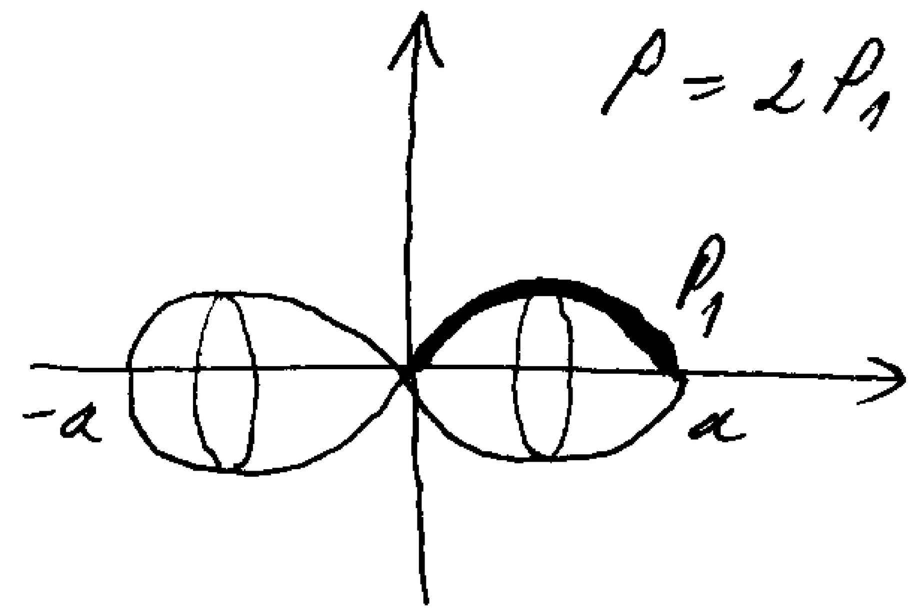
$$\left(P_1=2\pi\int_\alpha^\beta |y(t)|\sqrt{(x'(t))^2+(y'(t))^2}\,dt\right)$$

$$P=2P_1=2\cdot 2\pi\int_0^{\pi/2}|a\sin^3 t|\sqrt{9a^2\sin^2 t\cos^2 t}\,dt=\left\{t\Big|_0^{\pi/2}:\ \begin{matrix}\sin t\geq 0\\ \cos t\geq 0\end{matrix}\right\}$$

$$=4\pi\cdot 3a^2\int_0^{\pi/2}\sin^3 t\cdot\sin t\cdot\cos t\,dt=12a^2\pi\int_0^{\pi/2}\sin^4 t\cos t\,dt=$$

$$=\left\{\begin{matrix}\sin t=s\\ \cos t\,dt=ds\end{matrix}\right\}=12a^2\pi\int_0^1 s^4\,ds=12a^2\pi\left[\frac{s^5}{5}\right]_0^1=\frac{12a^2\pi}{5}$$

(*) Израчунати површину тела које настаје ротацијом око x-осе криве $\underbrace{(x^2+y^2)^2 = a^2(x^2-y^2)}_{\text{ЛЕМНИСКАТА}}$ $(a>0)$.



$P = 2P_1$

$$x = \rho\cos\varphi, \quad y = \rho\sin\varphi$$

$$(\rho^2)^2 = a^2(\rho^2\cos^2\varphi - \rho^2\sin^2\varphi)$$

$$\rho^4 = a^2\rho^2(\cos^2\varphi - \sin^2\varphi)$$

$$\rho^2 = a^2\cos 2\varphi$$

$$\rho = a\sqrt{\cos 2\varphi}$$

$$\rho' = a\frac{-2\sin 2\varphi}{2\sqrt{\cos 2\varphi}}$$

$$\rho^2 + \rho'^2 = a^2\cos 2\varphi + a^2\frac{\sin^2 2\varphi}{\cos 2\varphi} = a^2\frac{\cos^2 2\varphi + \sin^2 2\varphi}{\cos 2\varphi} = \frac{a^2}{\cos 2\varphi}$$

1. квадрант: $\varphi\Big|_0^{\pi/2}$

$\cos 2\varphi \geq 0: \quad 2\varphi\Big|_{-\pi/2}^{\pi/2}, \quad \varphi\Big|_{-\pi/4}^{\pi/4} \Big\} \quad \varphi\Big|_0^{\pi/4}$

$$P = 2P_1 = 2\cdot 2\pi\int_\alpha^\beta \rho|\sin\varphi|\sqrt{(\rho'(\varphi))^2 + (\rho(\varphi))^2}\,d\varphi =$$

$$= 4\pi\int_0^{\pi/4} a\cancel{\sqrt{\cos 2\varphi}}\underbrace{}_{}|\sin\varphi|\sqrt{\frac{a^2}{\cancel{\cos 2\varphi}}}\,d\varphi = 4\pi\cdot a^2\int_0^{\pi/4}\sin\varphi\,d\varphi =$$

$$= 4a^2\pi\left[-\cos\varphi\right]\Big|_0^{\pi/4} = -4a^2\pi\left(\frac{1}{\sqrt{2}} - 1\right) = \frac{\sqrt{2}-1}{\sqrt{2}}\cdot 4a^2\pi = 2a^2\pi(2-\sqrt{2})$$