Машински факултет
УНИВЕРЗИТЕТА У БЕОГРАДУ

ОДАБРАНА ПОГЛАВЉА ИЗ МЕХАНИКЕ

ДОКТОРСКЕ СТУДИЈЕ

ОСНОВЕ МЕХАНИКЕ КРУТИХ ТЕЛА

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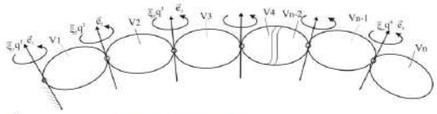
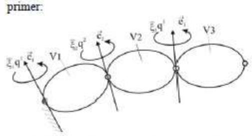
3.1Složene matrice transformacija



q^i, i=1,2,...,n- relativne koordinate osim prve

primer:

primer:



zaustavlja se 1 i 2 segment a treći rotira oko sada nepokretne ose 3 za ugao q^3 i tako redom.Sistem je u početnom trenutku nalazio u referentnom položaju.

$(V_3(0)) \xrightarrow{\bar{\xi}_3 q^3} (V_3(I))$,

$(V_3(I)) \xrightarrow{\bar{\xi}_2 q^2} (V_3(II))$, $(V_3(0))$-referentni položaj, (V_3)-proizvoljni položaj,

$(V_3(II)) \xrightarrow{\bar{\xi}_1 q^1} (V_3)$,

Interesuju nas projekcije uočenog vektora na ose Ox,Oz,Oy (koje odgovaraju referentnoj konfiguraciji i to posle rotacije)

$$\{\bar{\rho}_3\} = \begin{Bmatrix} \xi_3(0) \\ \eta_3(0) \\ \varsigma_3(0) \end{Bmatrix} \text{-referentna konfiguracija}$$

prva rotacija $\bar{\xi}_3 q^3$

$$\begin{Bmatrix} x' \\ y' \\ z' \end{Bmatrix} = \left[A_{R_3}\right] \begin{Bmatrix} \xi_3(0) \\ \eta_3(0) \\ \varsigma_3(0) \end{Bmatrix}$$

druga rotacija $\bar{\xi}_2 q^2$

$$\begin{Bmatrix} x'' \\ y'' \\ z'' \end{Bmatrix} = \left[A_{R_2}\right] \begin{Bmatrix} x' \\ y' \\ z' \end{Bmatrix} = \left[A_{R_2}\right]\left[A_{R_3}\right] \begin{Bmatrix} \xi_3(0) \\ \eta_3(0) \\ \varsigma_3(0) \end{Bmatrix}$$

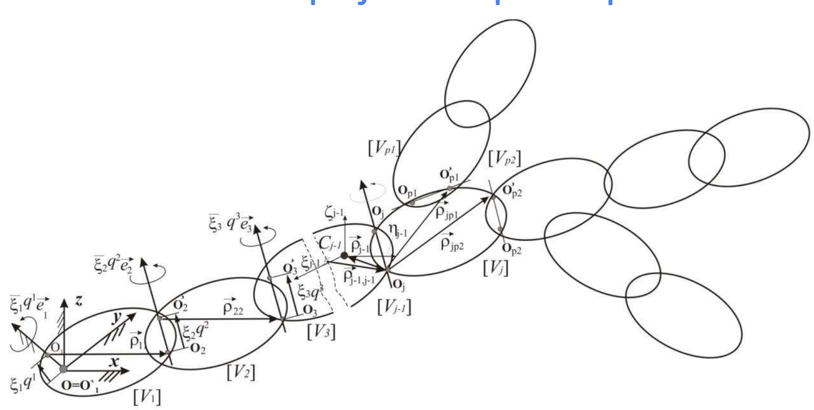
treća rotacija $\bar{\xi}_1 q^1$

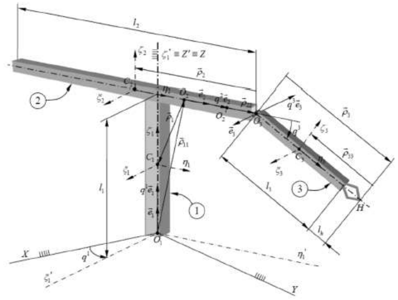
$$\begin{Bmatrix} x''' = x_3 \\ y''' = y_3 \\ z''' = z_3 \end{Bmatrix} = \left[A_{R_1}\right] \begin{Bmatrix} x'' \\ y'' \\ z'' \end{Bmatrix} = \left[A_{R_1}\right]\left[A_{R_2}\right]\left[A_{R_3}\right] \begin{Bmatrix} \xi_3(0) \\ \eta_3(0) \\ \varsigma_3(0) \end{Bmatrix} = \left[A_{0,3}\right] \begin{Bmatrix} \xi_3 \\ \eta_3 \\ \varsigma_3 \end{Bmatrix}$$

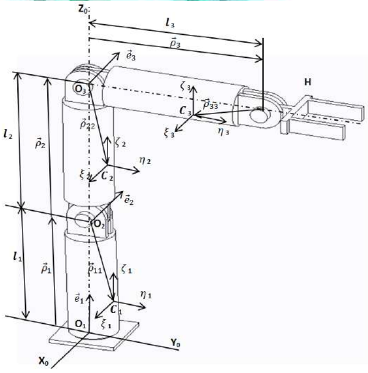
$\left[A_{0,3}\right]$ složena matrica transformacije

$$\left[A_{0,3}\right] = \left[A_{R_1}\right]\left[A_{R_2}\right]\left[A_{R_3}\right]$$

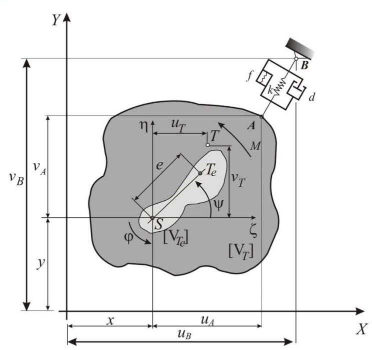
Основне геометријске карактеристике



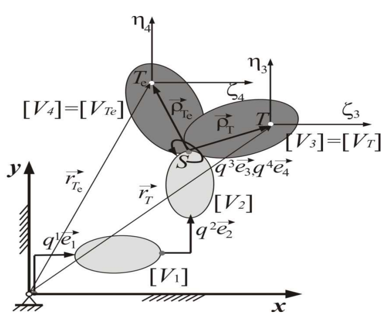




2D model of washing machine



. 2D model - classical approach



2D model - robotically approach

4 (DOFs) $q^1 = x, q^2 = y, q^3 = \varphi, q^4 = \psi,$

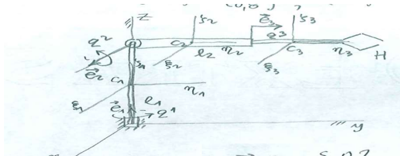
position [VT] $\overrightarrow{r_T} = q^1\vec{e}_1 + q^2\vec{e}_2 + \vec{\rho}_{33T} + \vec{\rho}_{3T}$

position [VTe] $\overrightarrow{r_{Te}} = q^1\vec{e}_1 + q^2\vec{e}_2 + \vec{\rho}_{33Te} + \vec{\rho}_{3Te}$

kinetic energy WM $E_k = \frac{1}{2}m\left(\dot{x}_T^2 + \dot{y}_T^2\right) + \frac{1}{2}J_T\left(\dot{q}^3\right)^2 + \frac{1}{2}m\left(\dot{x}_{Te}^2 + \dot{y}_{Te}^2\right) + \frac{1}{2}J_{Te}\left(\dot{q}^4\right)^2$

За дати роботски систем са три степена слободе који се налази у референтном положају, решити директни кинематички задатак, тј. одредити спољашње координате врха хватаљке $\bar{q}^i$ ($i=1,2,3$ – базни случај позиционирања) ако су вредности унутрашњих координата познате: $q^1=0,2$ rad, $q^2=0,4$ rad, $q^3=0,6$ m. Такође је познато:

$$\{\vec{\rho}_{11}\} = \begin{Bmatrix} 0 \\ 0 \\ 0,8 \end{Bmatrix}, \quad \{\vec{\rho}_{22}\} = \begin{Bmatrix} 0 \\ 0,6 \\ 0 \end{Bmatrix}, \quad \{\vec{\rho}_{33}\} = \begin{Bmatrix} 0 \\ 0,4 \\ 0 \end{Bmatrix}.$$



$\bar{q}^1 = x_H$, $\bar{q}^2 = y_H$, $\bar{q}^3 = z_H$ → спољашње координате које одређују позицију врха хватаљке у односу на непокретни систем Oxyz!

$$\{\vec{e}_1\} = \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix}, \quad \{\vec{e}_2\} = \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}, \quad \{\vec{e}_3\} = \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix}$$

Положај врха хватаљке у систему Oxyz:

$$\begin{Bmatrix} x_H \\ y_H \\ z_H \end{Bmatrix} = \sum_{j=1}^{3} [A_{0,j}] \left(\{\vec{\rho}_{jj} + \xi_j q^j \vec{e}_j\} \right) = [A_{0,1}]\{\vec{\rho}_{11}\} + [A_{0,2}]\{\vec{\rho}_{22}\} + [A_{0,3}]\left(\{\vec{\rho}_{33} + q^3 \vec{e}_3\}\right)$$

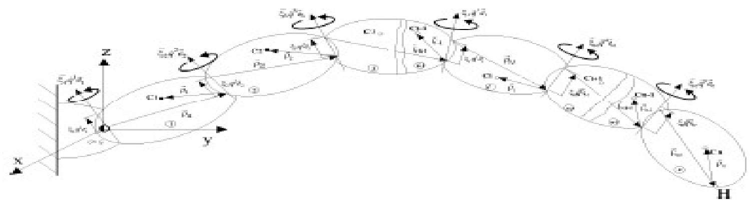
$$[A_{0,1}] = \begin{bmatrix} \cos q^1 & -\sin q^1 & 0 \\ \sin q^1 & \cos q^1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad [A_{1,2}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos q^2 & -\sin q^2 \\ 0 & \sin q^2 & \cos q^2 \end{bmatrix}, \quad [A_{2,3}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[A_{0,2}] = [A_{0,1}][A_{1,2}] = \begin{bmatrix} \cos q^1 & -\sin q^1 \cos q^2 & \sin q^1 \sin q^2 \\ \sin q^1 & \cos q^1 \cos q^2 & -\cos q^1 \sin q^2 \\ 0 & \sin q^2 & \cos q^2 \end{bmatrix}$$

$$[A_{0,3}] = [A_{0,2}][A_{2,3}] = [A_{0,2}][I] = [A_{0,2}]$$

$$\begin{Bmatrix} x_H \\ y_H \\ z_H \end{Bmatrix} = \begin{Bmatrix} -(1+q^3)\sin q^1 \cos q^2 \\ (1+q^3)\cos q^1 \cos q^2 \\ 0,8 + (1+q^3)\sin q^2 \end{Bmatrix}_{\substack{q^1=0,2 \\ q^2=0,4 \\ q^3=0,6}} = \begin{Bmatrix} -0,3 \\ 1,44 \\ 1,42 \end{Bmatrix} [m].$$

Угаона брзина $\vec{\omega}_i$



$$\vec{\omega}_1 = \bar{\xi}_1 \vec{e}_1 \dot{q}^1$$

$$\vec{\omega}_{2r} = \bar{\xi}_2 \vec{e}_2 \dot{q}^2,$$

$$\vec{\omega}_2 = \vec{\omega}_{2p} + \vec{\omega}_{2r}. \qquad \vec{\omega}_{2p} = \vec{\omega}_1,$$

$$\vec{\omega}_2 = \bar{\xi}_1 \vec{e}_1 \dot{q}^1 + \bar{\xi}_2 \vec{e}_2 \dot{q}^2.$$

$$\vec{\omega}_{3p} = \vec{\omega}_2, \quad \vec{\omega}_{3r} = \bar{\xi}_3 \vec{e}_3 \dot{q}^3,$$

$$\vec{\omega}_3 = \sum_{k=1}^{3} \bar{\xi}_k \vec{e}_k \dot{q}^k.$$

$$\vec{\omega}_i = \sum_{k=1}^{i} \bar{\xi}_k \vec{e}_k \dot{q}^k$$

$$\vec{\omega}_i = \sum_{k=1}^{n} \bar{\Omega}_{k(i)} \dot{q}^k,$$

$$\left\{\vec{\omega}_i^{(0)}\right\} = \sum_{k=1}^{i} \bar{\xi}_k \left[A_{0,k}\right] \left\{\vec{e}_k\right\} \dot{q}^k,$$

$$\left\{\vec{\omega}_i^{(0)}\right\} = [F]\{\dot{q}\},$$

gde je

$$[F] \in R^{3\times n} \Rightarrow [F] = \left[\bar{\xi}_1 \left\{\vec{e}_1^{(0)}\right\} \vdots \bar{\xi}_2 \left\{\vec{e}_2^{(0)}\right\} \vdots \ldots \vdots \bar{\xi}_i \left\{\vec{e}_i^{(0)}\right\}\right],$$

Извод вектора везаног за сегмент по генералисаној координати

$$\vec{\rho}_{ii} = \vec{\rho}_{ii}\left(q^1, q^2, \ldots, q^n\right).$$

$$\frac{\partial \vec{\rho}_{ii}}{\partial q^\alpha} = \bar{\xi}_\alpha \vec{e}_\alpha \times \vec{\rho}_{ii} \quad \forall \alpha \le i,$$

$$\frac{\partial \vec{\rho}_{ii}}{\partial q^\alpha} = 0 \quad \forall \alpha > i.$$

- Ојлеров образац

$$\frac{d\vec{\rho}_{ii}}{dt} = \vec{\omega}_i \times \vec{\rho}_{ii} = \sum_{\alpha=1}^{i} \bar{\xi}_\alpha \vec{e}_\alpha \dot{q}^\alpha \times \vec{\rho}_{ii} = \sum_{\alpha=1}^{i} \left(\bar{\xi}_\alpha \vec{e}_\alpha \times \vec{\rho}_{ii}\right) \dot{q}^\alpha$$

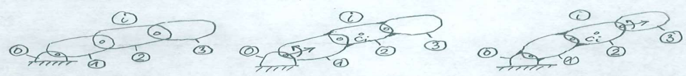
$$\frac{d\vec{\rho}_{ii}}{dt} = \sum_{\alpha=1}^{i} \frac{\partial \vec{\rho}_{ii}}{\partial q^\alpha} \dot{q}^\alpha = \sum_{\alpha=1}^{n} \frac{\partial \vec{\rho}_{ii}}{\partial q^\alpha} \dot{q}^\alpha, \quad \text{jer je} \quad \frac{\partial \vec{\rho}_{ii}}{\partial q^\alpha} = 0, \ \forall \alpha > i$$

$$0 = \sum_{\alpha=1}^{n} \left(\frac{\partial \vec{\rho}_{ii}}{\partial q^\alpha} - \bar{\xi}_\alpha \vec{e}_\alpha \times \vec{\rho}_{ii}\right) \dot{q}^\alpha \Rightarrow \begin{cases} \dfrac{\partial \vec{\rho}_{ii}}{\partial q^\alpha} = \bar{\xi}_\alpha \vec{e}_\alpha \times \vec{\rho}_{ii} & \forall \alpha \le i \\ \dfrac{\partial \vec{\rho}_{ii}}{\partial q^\alpha} = 0, & \forall \alpha > i \end{cases}$$

$$\left[|\vec{r}| = \text{const.} \rightarrow \frac{d\vec{r}}{d\varphi} = \vec{e} \times \vec{r}, \ \left(d\vec{r} = \frac{\partial \vec{r}}{\partial \varphi} d\varphi \rightarrow \frac{d\vec{r}}{d\varphi} = \frac{\partial \vec{r}}{\partial \varphi}\right) \Rightarrow \frac{\partial \vec{r}}{\partial \varphi} = \vec{e} \times \vec{r} \right]$$

* квазибазни вектори

$$\vec{T}_{i(\alpha)} = \frac{\partial \vec{r}_i}{\partial q^\alpha} = 0 \quad \text{за} \quad \alpha > i$$



$q^1=0,\ q^2=0,\ q^3=0$ $\qquad$ $q^1\neq 0,\ q^2=0,\ q^3=0$ $\qquad$ $q^1=0,\ q^2=0,\ q^3\neq 0$

Закључак: на промену вектора положаја $\vec{r}_i$ (тј. вектора положаја центра инерције C_i) утичу само тела која претходе сегменту (телу) са фиксираним индексом ⓘ а тела која следе иза тела са индексом ⓘ не утичу на промену вектора положаја $\vec{r}_i$, односно:

$$\vec{T}_{i(\alpha)} = \frac{\partial \vec{r}_i}{\partial q^\alpha} \neq 0 \quad \text{за} \quad \alpha \le i \quad ; \quad \vec{T}_{i(\alpha)} = \frac{\partial \vec{r}_i}{\partial q^\alpha} = 0 \quad \text{за} \quad \alpha > i \quad !$$

($\xi_\alpha = 1,\ \bar{\xi}_\alpha = 0 \Rightarrow$ транслаторни зглоб; $\xi_\alpha = 0,\ \bar{\xi}_\alpha = 1 \Rightarrow$ ротациони зглоб)

Брзина центра инерције крутог тела $\vec{v}_{C_i}$

$$\vec{v}_{C_i} = \vec{v}_i = \sum_{\alpha=1}^{i} \vec{T}_{\alpha(i)} \dot{q}^\alpha$$

$$\vec{T}_{\alpha(i)} = \partial \vec{r}_i / \partial q^\alpha$$

$$\vec{R}_{\alpha(i)} = \left[\sum_{k=\alpha}^{i} \left(\vec{\rho}_{kk} + \xi_k \vec{e}_k q^k \right) + \vec{\rho}_i \right]$$

$$\vec{T}_{\alpha(i)} = \bar{\xi}_\alpha \vec{e}_\alpha \times \vec{R}_{\alpha(i)} + \xi_\alpha \vec{e}_\alpha, \ \forall \alpha \le i$$
$$\vec{T}_{\alpha(i)} = 0, \ \forall \alpha > i,$$

$$\frac{\partial \vec{\rho}_{kk}}{\partial q^\alpha} = \bar{\xi}_\alpha \vec{e}_\alpha \times \vec{\rho}_{kk} \quad \forall \alpha \le k,$$
$$\frac{\partial \vec{\rho}_{kk}}{\partial q^\alpha} = 0 \quad \forall \alpha > k.$$
$$\frac{\partial \vec{e}_k}{\partial q^\alpha} = \bar{\xi}_\alpha \vec{e}_\alpha \times \vec{e}_k \quad \forall \alpha \le k,$$
$$\frac{\partial \vec{e}_k}{\partial q^\alpha} = 0 \quad \forall \alpha > k$$

Матрични облик

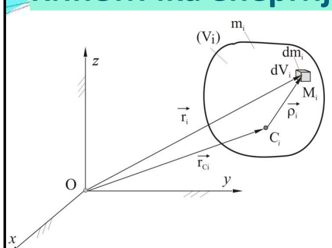
$$\left\{\vec{v}_i^{(0)}\right\} = \sum_{\alpha=1}^{n} \left\{\vec{T}_{\alpha(i)}^{(0)}\right\} \dot{q}^{\alpha}$$

$$\left\{\vec{T}_{\alpha(i)}^{(0)}\right\} = \bar{\xi}_\alpha \left[A_{0,\alpha}\right]\left[e_\alpha^d\right]\left\{\sum_{k=\alpha}^{i}\left[A_{\alpha,k}\right]\left(\left\{\vec{\rho}_{kk}\right\} + \bar{\xi}_k q^k \left\{\vec{e}_k\right\}\right) + \left[A_{\alpha,i}\right]\left\{\vec{\rho}_k\right\}\right\} +$$
$$+ \xi_\alpha \left[A_{0,\alpha}\right]\left\{\vec{e}_\alpha\right\} \quad \forall \alpha \le i,$$
$$\left\{\vec{T}_{\alpha(i)}^{(0)}\right\} = 0 \quad \forall \alpha > i.$$

$$\left[E\right] \in R^{3\times n} \Rightarrow \left[E\right] = \left[\left\{\vec{T}_{1(i)}^{(0)}\right\} \vdots \left\{\vec{T}_{2(i)}^{(0)}\right\} \vdots \ldots \vdots \left\{\vec{T}_{n(i)}^{(0)}\right\}\right]$$

$$\left\{\vec{v}_i^{(0)}\right\} = \left[E\right]\left\{\dot{q}\right\},$$

Кинетичка енергија [Vi] роботског сегмента и РС



$$dE_{k(i)} = \frac{1}{2} dm_i\, v_{M_i}^2 \qquad \vec{v}_{M_i} = \vec{v}_i = \vec{v}_{C_i} + \vec{\omega}_i \times \vec{\rho}_i$$

$$E_{k(i)} = \frac{1}{2}\int_{(V_i)} \left(\vec{v}_{C_i} + \vec{\omega}_i \times \vec{\rho}_i\right)\cdot\left(\vec{v}_{C_i} + \vec{\omega}_i \times \vec{\rho}_i\right) dm_i$$

$$E_{k(i)} = \frac{1}{2} v_{C_i}^2 \int_{(V_i)} dm_i + \left(\vec{v}_{C_i} \times \vec{\omega}_i\right)\cdot \int_{(V_i)} \vec{\rho}_i dm_i + \frac{1}{2}\int_{(V_i)} \left(\vec{\omega}_i \times \vec{\rho}_i\right)\cdot\left(\vec{\omega}_i \times \vec{\rho}_i\right) dm_i$$

$$(\vec{\omega}_i \times \vec{\rho}^{(i)})\cdot(\vec{\omega}_i \times \vec{\rho}^{(i)}) = (\vec{\omega}_i \times \vec{\rho}^{(i)})\{\vec{\omega}_i \times \vec{\rho}^{(i)}\} \rightarrow \{\vec{\omega}_i \times \vec{\rho}^{(i)}\} = -[\rho^{(i)d}]\{\vec{\omega}_i\}$$

$$(\vec{\omega}_i \times \vec{\rho}^{(i)}) = \{\vec{\omega}_i \times \vec{\rho}^{(i)}\}^T = (-[\rho^{(i)d}]\{\vec{\omega}_i\})^T = -\{\vec{\omega}_i\}^T[\rho^{(i)d}]^T = -(\vec{\omega}_i)(-[\rho^{(i)d}]) = (\vec{\omega}_i)[\rho^{(i)d}]$$

$$(\vec{\omega}_i \times \vec{\rho}^{(i)})\cdot(\vec{\omega}_i \times \vec{\rho}^{(i)}) = -(\vec{\omega}_i)[\rho^{(i)d}][\rho^{(i)d}]\{\vec{\omega}_i\} = -(\vec{\omega}_i)[\rho^{(i)d}]^2\{\vec{\omega}_i\}$$

$$E_{k(i)} = \frac{1}{2} m_i v_{C_i}^2 + \frac{1}{2}\int_{(V_i)} \left(\vec{\omega}_i \times \vec{\rho}_i\right)\cdot\left(\vec{\omega}_i \times \vec{\rho}_i\right) dm_i$$

$$\left(\vec{\omega}_i \times \vec{\rho}_i\right)\cdot\left(\vec{\omega}_i \times \vec{\rho}_i\right) = \left(\vec{\omega}_i \times \vec{\rho}_i\right)\cdot\left\{\vec{\omega}_i \times \vec{\rho}_i\right\} = -\left(\vec{\omega}_i\right)\left[\rho_i^d\right]^2\left\{\vec{\omega}_i\right\}$$

$$\left[J_A\right] \overset{def}{=} -\int_{(V)} \left[\rho^d\right]^2 dm \qquad \left[J_A\right] = \begin{bmatrix} \int_{(v)} (\eta^2+\zeta^2)dm & -\int_{(v)} (\xi\eta)dm & -\int_{(v)} (\xi\zeta)dm \\ -\int_{(v)} (\eta\xi)dm & \int_{(v)} (\zeta^2+\xi^2)dm & -\int_{(v)} (\eta\zeta)dm \\ -\int_{(v)} (\zeta\xi)dm & -\int_{(v)} (\zeta\eta)dm & \int_{(v)} (\eta^2+\xi^2)dm \end{bmatrix}$$

$$E_{k(i)} = \frac{1}{2} m_i v_{C_i}^2 + \frac{1}{2}\left(\vec{\omega}_i\right)\left[J_{C_i}\right]\left\{\vec{\omega}_i\right\}$$

$$\vec{\rho}^{(i)} = \xi^{(i)}\vec{\lambda} + \eta^{(i)}\vec{\mu} + \zeta^{(i)}\vec{\nu}$$

$$-[\rho^{(i)d}]^2 = \begin{bmatrix} \eta^{(i)2}+\zeta^{(i)2} & -\xi^{(i)}\eta^{(i)} & -\xi^{(i)}\zeta^{(i)} \\ -\xi^{(i)}\eta^{(i)} & \xi^{(i)2}+\zeta^{(i)2} & -\eta^{(i)}\zeta^{(i)} \\ -\xi^{(i)}\zeta^{(i)} & -\eta^{(i)}\zeta^{(i)} & \xi^{(i)2}+\eta^{(i)2} \end{bmatrix} \Rightarrow \int_{(V_i)} -[\rho^{(i)d}]^2 dm = \begin{bmatrix} J_{\xi_i} & -J_{\xi_i\eta_i} & -J_{\xi_i\zeta_i} \\ -J_{\xi_i\eta_i} & J_{\eta_i} & -J_{\eta_i\zeta_i} \\ -J_{\xi_i\zeta_i} & -J_{\eta_i\zeta_i} & J_{\zeta_i} \end{bmatrix} = [$$

Ектр Кинетичка енергија транслације

$$E_{k(i)tr} = \frac{1}{2} m_i v_{C_i}^2$$

$$\vec{v}_{C_i} = \sum_{\alpha=1}^{n} \vec{T}_{\alpha(i)} \dot{q}^{\alpha}$$

$$E_k^{(tr)} = \sum_{i=1}^{n} E_k^{(i,tr)} = \frac{1}{2}\sum_{i=1}^{n} m_i v_i^2 = \frac{1}{2}\sum_{i=1}^{n} m_i \vec{v}_i \cdot \vec{v}_i$$

$$\vec{v}_i (\equiv \vec{v}_{C_i}) = \sum_{\alpha=1}^{n} \frac{\partial \vec{r}_i}{\partial q^{\alpha}} \dot{q}^{\alpha} \rightarrow E_k^{(tr)} = \frac{1}{2}\sum_{i=1}^{n} m_i \left(\sum_{\alpha=1}^{n} \frac{\partial \vec{r}_i}{\partial q^{\alpha}} \dot{q}^{\alpha}\right) \cdot \left(\sum_{\beta=1}^{n} \frac{\partial \vec{r}_i}{\partial q^{\beta}} \dot{q}^{\beta}\right)$$

$$a_{\alpha\beta}^{(tr)} = \sum_{i=1}^{n} m_i \frac{\partial \vec{r}_i}{\partial q^{\alpha}} \cdot \frac{\partial \vec{r}_i}{\partial q^{\beta}} \rightarrow a_{\alpha\beta}^{(tr)} = a_{\beta\alpha}^{(tr)} \; ; \quad \vec{T}_{i(\alpha)} = \frac{\partial \vec{r}_i}{\partial q^{\alpha}} \Rightarrow$$

$$a_{\alpha\beta}^{(tr)} = \sum_{i=1}^{n} m_i \vec{T}_{i(\alpha)} \cdot \vec{T}_{i(\beta)}$$

$$E_{ki} = \frac{1}{2} m_i \left(\sum_{\alpha=1}^{n} \vec{T}_{\alpha(i)} \dot{q}^{\alpha}\right) \cdot \left\{\sum_{\beta=1}^{n} \vec{T}_{\beta(i)} \dot{q}^{\beta}\right\} \Rightarrow E_{ki}tr = \frac{1}{2}\sum_{\alpha=1}^{n}\sum_{\beta=1}^{n} a^{tr}{}_{\alpha\beta(i)} \dot{q}^{\alpha} \dot{q}^{\beta}$$

$$a^{tr}{}_{\alpha\beta(i)} = m_i \left(\vec{T}_{\alpha(i)}\right)\left\{\vec{T}_{\beta(i)}\right\}$$

Екрот Кинетичка енергија ротације

$$E_{k(i)rot} = \frac{1}{2}\left(\vec{\omega}_i\right)\left[J_{C_i}\right]\left\{\vec{\omega}_i\right\} \Leftarrow \vec{\omega}_i = \sum_{\alpha=1}^{n} \vec{\Omega}_{\alpha(i)} \dot{q}^{\alpha}$$

$$E_{ki}rot = \frac{1}{2}\left(\sum_{\alpha=1}^{n} \vec{\Omega}_{\alpha(i)} \dot{q}^{\alpha}\right)\left[J_{C_i}\right]\left\{\sum_{\beta=1}^{n} \vec{\Omega}_{\beta(i)} \dot{q}^{\beta}\right\},$$

$$E_{ki}rot = \frac{1}{2}\sum_{\alpha=1}^{n}\sum_{\beta=1}^{n} a^{rot}{}_{\alpha\beta(i)} \dot{q}^{\alpha} \dot{q}^{\beta}$$

$$a^{rot}{}_{\alpha\beta(i)} = \left(\vec{\Omega}_{\alpha(i)}\right)\left[J_{C_i}\right]\left\{\vec{\Omega}_{\beta(i)}\right\}$$

$$E_{ki} = \frac{1}{2}\sum_{\alpha=1}^{n}\sum_{\beta=1}^{n} a_{\alpha\beta(i)} \dot{q}^{\alpha} \dot{q}^{\beta}$$

Ek роботског система

$$E_k = \sum_{i=1}^{n} E_{k(i)}$$

$$E_k = \frac{1}{2}\sum_{\alpha=1}^{n}\sum_{\beta=1}^{n}\sum_{i=1}^{n} a_{\alpha\beta(i)}\dot{q}^{\alpha}\dot{q}^{\beta}$$

$(a_{\alpha\beta} = a^{(tr)}_{\alpha\beta} + a^{(rot)}_{\alpha\beta})$

$$E_k = \frac{1}{2}\sum_{\alpha=1}^{n}\sum_{\beta=1}^{n} a_{\alpha\beta}\dot{q}^{\alpha}\dot{q}^{\beta}$$

$$a_{\alpha\beta} = \sum_{i=1}^{n} m_i\left(\vec{T}_{\alpha(i)}\right)\left\{\vec{T}_{\beta(i)}\right\} + \sum_{i=1}^{n}\left(\vec{\Omega}_{\alpha(i)}\right)[J_{Ci}]\left\{\vec{\Omega}_{\beta(i)}\right\}$$

$$a_{\alpha\beta} = a_{\beta\alpha}$$

- Основни метрички тензор

$$a_{\alpha\beta} = \sum_{i=1}^{n} m_i \frac{\partial \vec{r}_{C_i}}{\partial q^{\alpha}}\cdot\frac{\partial \vec{r}_{C_i}}{\partial q^{\beta}} + \sum_{i=1}^{n}\int_{(Vi)}\frac{\partial \vec{\rho}_i}{\partial q^{\alpha}}\cdot\frac{\partial \vec{\rho}_i}{\partial q^{\beta}}dm_i$$

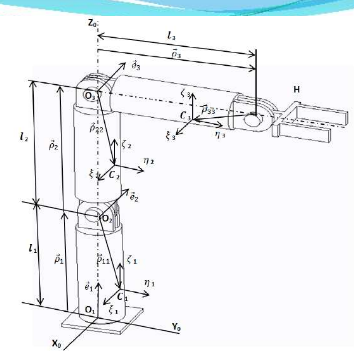
$$[a_{\alpha\beta}] \in R^{n\times n}$$

$$\frac{\partial \vec{r}_{C_i}}{\partial q^{\alpha}} = \vec{T}_{\alpha(i)},\ \frac{\partial \vec{\rho}_i}{\partial q^{\alpha}} = \vec{\Omega}_{\alpha(i)}\times\vec{\rho}_i,\ \left[J_{C_i}\right] = -\int_{(Vi)}\left[\rho_i^d\right]^2 dm_i$$

$$E_k = \frac{1}{2}(\dot{q})\left[a_{\alpha\beta}\right]\{\dot{q}\}$$

- $E_k(\sum_{i=1}^{n}(\dot{q}^i)^2 \neq 0) > 0, E_k(\sum_{i=1}^{n}(\dot{q}^i)^2 = 0) = 0,$
- pozitivno definitna kvadratna forma generalisanih brzina.

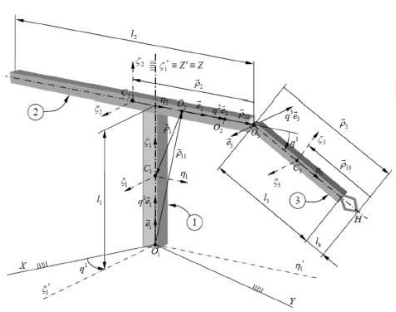
Пример РС- 3сс



$$\mathrm{A}(\boldsymbol{q}) = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$a_{\alpha\beta} = \sum_{i=\sup(\alpha,\beta)}^{n} m_i(\vec{T}_{\alpha(i)})\{\vec{T}_{\beta(i)}\} + \sum_{i=\sup(\alpha,\beta)}^{n} \bar{\xi}_{\alpha}\bar{\xi}_{\beta}(\vec{e}_{\alpha(i)})[J_{ci}]\{\vec{e}_{\beta(i)}\}$$

$$a_{13} = m_3(\vec{T}_{1(3)})\{\vec{T}_{3(3)}\} + \bar{\xi}_1\bar{\xi}_3([\mathrm{A}_{13}]^{\mathrm{T}}\{\vec{e}_{1(3)}\})[J_{c3}]\{\vec{e}_{3(3)}\}$$



$$a_{11} = \sum_{i=1}^{3} a^{tr}_{11,i} + a^{rot}_{11,i} = m_2\left(\frac{l_2}{2} - d_1 - q^2\right)^2 + m_3\left(d_1 + q^2 + \frac{l_3}{2}\cos q^3\right)^2 + + J_{\zeta_1} + J_{\zeta_2} + \frac{1}{2}\left(J_{\eta_3} + J_{\zeta_3}\right) + \frac{1}{2}\left(J_{\zeta_3} - J_{\eta_3}\right)\cos 2q^3$$

$$a_{12} = \sum_{i=1}^{3} a^{tr}_{12,i} + a^{rot}_{12,i} = 0,\ a_{22} = \sum_{i=1}^{3} a^{tr}_{22,i} + a^{rot}_{22,i} = m_2 + m_3,\ a_{13} = \sum_{i=1}^{3} a^{tr}_{13,i} + a^{rot}_{13,i} = 0$$

$$a_{23} = \sum_{i=1}^{3} a^{tr}_{23,i} + a^{rot}_{23,i} = -\frac{m_3 l_3}{2}\sin q^3,\ a_{33} = \sum_{i=1}^{3} a^{tr}_{33,i} + a^{rot}_{33,i} = \frac{1}{4}m_3 l_3^2 + J_{\xi_3}$$

$$A(\mathbf{q}) = \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & a_{23} \\ 0 & a_{23} & a_{33} \end{bmatrix}$$

```
Ek1= 0.5*m1*(Vc11'*Vc11)+ 0.5*w11'*J1*w11;
Ek2= 0.5*m2*(Vc21'*Vc21)+ 0.5*w22'*J2*w22;
Ek3= 0.5*m3*(Vc31'*Vc31)+ 0.5*w33'*J3*w33;
Ek=Ek1 + Ek2 + Ek3;
```

$$E_k = \frac{1}{2}\sum_{\alpha=1}^{n}\sum_{\beta=1}^{n} a_{\alpha\beta}\dot{q}^{\alpha}\dot{q}^{\beta}$$

```
%Koeficijenti metrickog tenzora
a11=diff(diff(Ek,q1d),q1d);
a12=diff(diff(Ek,q1d),q2d);
a13=diff(diff(Ek,q1d),q3d);
a22=diff(diff(Ek,q2d),q2d);
a23=diff(diff(Ek,q2d),q3d);
a33=diff(diff(Ek,q3d),q3d);
a21=a12; a31=a13; a32=a23;
A=[a11 a12 a13; a21 a22 a23; a31 a32 a33];
```

$$a_{\alpha\beta} = \frac{\partial^2 E_k}{\partial \dot{q}^{\alpha}\partial \dot{q}^{\beta}}, \quad \Rightarrow a_{\alpha\beta} = a_{\beta\alpha}.$$

Коваријантни облик диференцијалних кретања РС

- Лагранжеве једначине друге врсте:

$(q^1, q^2, \ldots, q^n)$

$$\frac{d}{dt}\left(\frac{\partial E_k}{\partial \dot{q}^{\gamma}}\right) - \frac{\partial E_k}{\partial q^{\gamma}} = Q_{\gamma}, \quad \gamma = 1,2,\ldots,n$$

$$\frac{\partial E_k}{\partial \dot{q}^{\gamma}} = \frac{1}{2}\sum_{\beta=1}^{n} a_{\gamma\beta}\dot{q}^{\beta} + \frac{1}{2}\sum_{\alpha=1}^{n} a_{\alpha\gamma}\dot{q}^{\alpha} \qquad \frac{\partial E_k}{\partial \dot{q}^{\gamma}} = \sum_{\alpha=1}^{n} a_{\alpha\gamma}\dot{q}^{\alpha}$$

$$\sum_{\alpha=1}^{n}\sum_{\beta=1}^{n}\frac{\partial a_{\alpha\gamma}}{\partial q^{\beta}}\dot{q}^{\alpha}\dot{q}^{\beta} = \frac{1}{2}\sum_{\alpha=1}^{n}\sum_{\beta=1}^{n}\frac{\partial a_{\alpha\gamma}}{\partial q^{\beta}}\dot{q}^{\alpha}\dot{q}^{\beta} + \frac{1}{2}\sum_{\alpha=1}^{n}\sum_{\beta=1}^{n}\frac{\partial a_{\beta\gamma}}{\partial q^{\alpha}}\dot{q}^{\alpha}\dot{q}^{\beta})$$

$$\frac{d}{dt}\left(\frac{\partial E_k}{\partial \dot{q}^{\gamma}}\right) = \sum_{\alpha=1}^{n} a_{\alpha\gamma}\ddot{q}^{\alpha} + \sum_{\alpha=1}^{n}\sum_{\beta=1}^{n}\frac{\partial a_{\alpha\gamma}}{\partial q^{\beta}}\dot{q}^{\alpha}\dot{q}^{\beta}$$

$$\frac{d}{dt}\left(\frac{\partial E_k}{\partial \dot{q}^{\gamma}}\right) = \sum_{\alpha=1}^{n} a_{\alpha\gamma}\ddot{q}^{\alpha} + \frac{1}{2}\sum_{\alpha=1}^{n}\sum_{\beta=1}^{n}\left(\frac{\partial a_{\alpha\gamma}}{\partial q^{\beta}} + \frac{\partial a_{\beta\gamma}}{\partial q^{\alpha}}\right)\dot{q}^{\alpha}\dot{q}^{\beta}$$

- . $\dfrac{\partial E_k}{\partial q^{\gamma}} = \dfrac{1}{2}\displaystyle\sum_{\alpha=1}^{n}\sum_{\beta=1}^{n}\frac{\partial a_{\alpha\gamma}}{\partial q^{\beta}}\dot{q}^{\alpha}\dot{q}^{\beta}$

$$\Gamma_{\alpha\beta,\gamma} = \frac{1}{2}\left(\frac{\partial a_{\beta\gamma}}{\partial q^{\alpha}} + \frac{\partial a_{\gamma\alpha}}{\partial q^{\beta}} - \frac{\partial a_{\alpha\beta}}{\partial q^{\gamma}}\right)$$

$$\sum_{\alpha=1}^{n} a_{\alpha\gamma}\ddot{q}^{\alpha} + \sum_{\alpha=1}^{n}\sum_{\beta=1}^{n}\Gamma_{\alpha\beta,\gamma}\dot{q}^{\alpha}\dot{q}^{\beta} = Q_{\gamma}$$

$$\Gamma_{\alpha\beta,\gamma} = \Gamma_{\beta\alpha,\gamma}, \quad \alpha,\beta,\gamma = 1,2,\ldots,n$$

- Лагранжеве једначине друге врсте - **коваријантни облик :**

$$\sum_{\alpha=1}^{n} a_{\alpha\gamma}\ddot{q}^{\alpha} + \sum_{\alpha=1}^{n}\sum_{\beta=1}^{n}\Gamma_{\alpha\beta,\gamma}\dot{q}^{\alpha}\dot{q}^{\beta} = Q_{\gamma}$$

- $a_{\gamma\alpha}$ - коефицијенти метричког тензора;
- $\Gamma_{\alpha\beta,\gamma}$- Кристофелови симболи прве врсте;
- Q_{γ} - генералисане силе система активних сила које делују на роботски систем

Матрична форма Лан. јед

$$\mathrm{K}_{\alpha\beta,\gamma} = \frac{1}{2}\left(\frac{\partial a_{\beta\gamma}}{\partial q_{\alpha}} + \frac{\partial a_{\alpha\gamma}}{\partial q_{\beta}} - \frac{\partial a_{\alpha\beta}}{\partial q_{\gamma}}\right)$$

$$A(\mathbf{q})\ddot{\mathbf{q}} + \mathrm{K}(\mathbf{q},\dot{\mathbf{q}}) = \mathbf{Q} \qquad A(\mathbf{q}) = \left[a_{\alpha\beta}\right]_{n\times n} \qquad K(\mathbf{q},\dot{\mathbf{q}}) = \left[\dot{\mathbf{q}}^{\mathrm{T}}\mathrm{K}_{1}\dot{\mathbf{q}},\ldots,\dot{\mathbf{q}}^{\mathrm{T}}\mathrm{K}_{n}\dot{\mathbf{q}}\right]$$

$K_{r} \rightarrow \gamma$

Кристофелови символи 1 врсте

$$\Gamma_{\alpha\beta,\gamma} = \frac{1}{2}\left(\frac{\partial a_{\beta\gamma}}{\partial q^{\alpha}} + \frac{\partial a_{\gamma\alpha}}{\partial q^{\beta}} - \frac{\partial a_{\alpha\beta}}{\partial q^{\gamma}}\right) \qquad \Gamma_{\alpha\beta,\gamma} = \Gamma_{\beta\alpha,\gamma}, \quad \alpha,\beta,\gamma = 1,2,\ldots,n$$

$$\Gamma_{\alpha\beta,\gamma} = -\Gamma_{\alpha\gamma,\beta}, \quad \alpha,\beta,\gamma = 1,2,\ldots,n$$

Симетричан по прва 2 индекса, и антисиметричан по друга 2 индекса

Аналитички израз за Кристофелов симбол 1 врсте

$$a_{\alpha\beta} = \sum_{i=1}^{n} m_i \frac{\partial\vec{r}_{C_i}}{\partial q^{\alpha}}\cdot\frac{\partial\vec{r}_{C_i}}{\partial q^{\beta}} + \sum_{i=1}^{n}\int_{(Vi)}\frac{\partial\vec{\rho}_i}{\partial q^{\alpha}}\cdot\frac{\partial\vec{\rho}_i}{\partial q^{\beta}}dm_i \qquad \frac{\partial\vec{r}_{C_i}}{\partial q^{\alpha}} = \vec{T}_{\alpha(i)}, \ \frac{\partial\vec{\rho}_i}{\partial q^{\alpha}} = \vec{\Omega}_{\alpha(i)}\times\vec{\rho}_i, \ \left[J_{C_i}\right] = -\int_{(Vi)}\left[\rho_i^{d}\right]^2 dm_i$$

$$\frac{\partial a_{\beta\gamma}}{\partial q^{\alpha}} = \sum_{i=1}^{n} m_i\left(\frac{\partial^2\vec{r}_{C_i}}{\partial q^{\alpha}\partial q^{\beta}}\cdot\frac{\partial\vec{r}_{C_i}}{\partial q^{\gamma}} + \frac{\partial\vec{r}_{C_i}}{\partial q^{\beta}}\cdot\frac{\partial^2\vec{r}_{C_i}}{\partial q^{\alpha}\partial q^{\gamma}}\right) + \sum_{i=1}^{n}\int_{(Vi)}\left(\frac{\partial^2\vec{\rho}_i}{\partial q^{\alpha}\partial q^{\beta}}\cdot\frac{\partial\vec{\rho}_i}{\partial q^{\gamma}} + \frac{\partial\vec{\rho}_i}{\partial q^{\beta}}\cdot\frac{\partial^2\vec{\rho}_i}{\partial q^{\alpha}\partial q^{\gamma}}\right)dm_i$$

$$\Gamma_{\alpha\beta,\gamma} = \frac{1}{2}\sum_{i=1}^{n} m_i\left(\begin{array}{l}\frac{\partial^2\vec{r}_{C_i}}{\partial q^{\alpha}\partial q^{\beta}}\cdot\frac{\partial\vec{r}_{C_i}}{\partial q^{\gamma}} + \frac{\partial\vec{r}_{C_i}}{\partial q^{\beta}}\cdot\frac{\partial^2\vec{r}_{C_i}}{\partial q^{\alpha}\partial q^{\gamma}} + \frac{\partial^2\vec{r}_{C_i}}{\partial q^{\beta}\partial q^{\gamma}}\cdot\frac{\partial\vec{r}_{C_i}}{\partial q^{\alpha}} \\ + \frac{\partial\vec{r}_{C_i}}{\partial q^{\gamma}}\cdot\frac{\partial^2\vec{r}_{C_i}}{\partial q^{\beta}\partial q^{\alpha}} - \frac{\partial^2\vec{r}_{C_i}}{\partial q^{\alpha}\partial q^{\gamma}}\cdot\frac{\partial\vec{r}_{C_i}}{\partial q^{\beta}} - \frac{\partial\vec{r}_{C_i}}{\partial q^{\alpha}}\cdot\frac{\partial^2\vec{r}_{C_i}}{\partial q^{\beta}\partial q^{\gamma}}\end{array}\right) + \frac{1}{2}\sum_{i=1}^{n}\int_{(Vi)}\left(\begin{array}{l}\frac{\partial^2\vec{\rho}_i}{\partial q^{\alpha}\partial q^{\beta}}\cdot\frac{\partial\vec{\rho}_i}{\partial q^{\gamma}} + \frac{\partial\vec{\rho}_i}{\partial q^{\beta}}\cdot\frac{\partial^2\vec{\rho}_i}{\partial q^{\alpha}\partial q^{\gamma}} + \frac{\partial^2\vec{\rho}_i}{\partial q^{\gamma}\partial q^{\beta}}\cdot\frac{\partial\vec{\rho}_i}{\partial q^{\alpha}} \\ + \frac{\partial\vec{\rho}_i}{\partial q^{\gamma}}\cdot\frac{\partial^2\vec{\rho}_i}{\partial q^{\alpha}\partial q^{\beta}} - \frac{\partial^2\vec{\rho}_i}{\partial q^{\alpha}\partial q^{\gamma}}\cdot\frac{\partial\vec{\rho}_i}{\partial q^{\beta}} - \frac{\partial\vec{\rho}_i}{\partial q^{\alpha}}\cdot\frac{\partial^2\vec{\rho}_i}{\partial q^{\beta}\partial q^{\gamma}}\end{array}\right)dm_i$$

$$\Gamma_{\alpha\beta,\gamma} = \sum_{i=1}^{n} m_i \frac{\partial^2 \vec{r}_{C_i}}{\partial q^\alpha \partial q^\beta} \cdot \frac{\partial \vec{r}_{C_i}}{\partial q^\gamma} + \sum_{i=1}^{n} \int_{(Vi)} \frac{\partial^2 \vec{\rho}_i}{\partial q^\alpha \partial q^\beta} \cdot \frac{\partial \vec{\rho}_i}{\partial q^\gamma} dm_i$$