Кристофелови симболи 1 врсте

$$\Gamma_{\alpha\beta,\gamma} = \frac{1}{2}\left(\frac{\partial a_{\beta\gamma}}{\partial q^{\alpha}} + \frac{\partial a_{\gamma_\alpha}}{\partial q^{\beta}} - \frac{\partial a_{\alpha\beta}}{\partial q^{\gamma}}\right)$$

$$\Gamma_{\alpha\beta,\gamma} = \Gamma_{\beta\alpha,\gamma}, \quad \alpha,\beta,\gamma = 1,2,\ldots,n$$

$$\Gamma_{\alpha\beta,\gamma} = -\Gamma_{\alpha\gamma,\beta}, \quad \alpha,\beta,\gamma = 1,2,\ldots,n$$

Симетричан по прва 2 индекса, и антисиметричан по друга 2 индекса

Аналитички израз за Кристофелов симбол 1 врсте

$$a_{\alpha\beta} = \sum_{i=1}^{n} m_i \frac{\partial \vec{r}_{C_i}}{\partial q^{\alpha}} \cdot \frac{\partial \vec{r}_{C_i}}{\partial q^{\beta}} + \sum_{i=1}^{n} \int_{(Vi)} \frac{\partial \vec{\rho}_i}{\partial q^{\alpha}} \cdot \frac{\partial \vec{\rho}_i}{\partial q^{\beta}} dm_i \qquad \frac{\partial \vec{r}_{C_i}}{\partial q^{\alpha}} = \vec{T}_{\alpha(i)}, \ \frac{\partial \vec{\rho}_i}{\partial q^{\alpha}} = \vec{\Omega}_{\alpha(i)} \times \vec{\rho}_i, \ \left[J_{C_i}\right] = -\int_{(Vi)} \left[\rho_i^d\right]^2 dm_i$$

$$\frac{\partial a_{\beta\gamma}}{\partial q^{\alpha}} = \sum_{i=1}^{n} m_i \left(\frac{\partial^2 \vec{r}_{C_i}}{\partial q^{\alpha}\partial q^{\beta}} \cdot \frac{\partial \vec{r}_{C_i}}{\partial q^{\gamma}} + \frac{\partial \vec{r}_{C_i}}{\partial q^{\beta}} \cdot \frac{\partial^2 \vec{r}_{C_i}}{\partial q^{\alpha}\partial q^{\gamma}} \right) + \sum_{i=1}^{n} \int_{(Vi)} \left(\frac{\partial^2 \vec{\rho}_i}{\partial q^{\alpha}\partial q^{\beta}} \cdot \frac{\partial \vec{\rho}_i}{\partial q^{\gamma}} + \frac{\partial \vec{\rho}_i}{\partial q^{\beta}} \cdot \frac{\partial^2 \vec{\rho}_i}{\partial q^{\alpha}\partial q^{\gamma}} \right) dm_i$$

$$\Gamma_{\alpha\beta,\gamma} = \sum_{i=1}^{n} m_i \frac{\partial^2 \vec{r}_{C_i}}{\partial q^{\alpha}\partial q^{\beta}} \cdot \frac{\partial \vec{r}_{C_i}}{\partial q^{\gamma}} + \sum_{i=1}^{n} \int_{(Vi)} \frac{\partial^2 \vec{\rho}_i}{\partial q^{\alpha}\partial q^{\beta}} \cdot \frac{\partial \vec{\rho}_i}{\partial q^{\gamma}} dm_i$$

$$\Gamma_{\alpha\beta,\gamma} = \Gamma^{tr}{}_{\alpha\beta,\gamma} + \Gamma^{rot}{}_{\alpha\beta,\gamma}$$

$$\Gamma^{tr}{}_{\alpha\beta,\gamma} = \sum_{i=1}^{n} m_i \frac{\partial^2 \vec{r}_{C_i}}{\partial q^{\alpha}\partial q^{\beta}} \frac{\partial \vec{r}_{C_i}}{\partial q^{\gamma}}, \quad \alpha \le \beta$$

$$\vec{T}_{\alpha(i)} = \frac{\partial \vec{r}_i}{\partial q^{\alpha}}, \quad \vec{T}_{\beta(i)} = \frac{\partial \vec{r}_i}{\partial q^{\beta}},$$

$$\vec{T}_{\alpha(i)} = \frac{\partial \vec{r}_i}{\partial q^{\alpha}} = \bar{\xi}_{\alpha}\vec{e}_{\alpha} \times \vec{R}_{\alpha(i)} + \xi_{\alpha}\vec{e}_{\alpha}$$

$$\vec{R}_{\alpha(i)} = \sum_{k=\alpha}^{i} \left(\vec{\rho}_{kk} + \xi_k \vec{e}_k q^k \right) + \vec{\rho}_i$$

$$\frac{\partial^2 \vec{r}_{C_i}}{\partial q^{\alpha}\partial q^{\beta}} = \frac{\partial \vec{T}_{\alpha(i)}}{\partial q^{\beta}}, \quad \alpha \le \beta$$

$$\frac{\partial \vec{r}_{ci}}{\partial q^{\gamma}} = \vec{T}_{\gamma(i)}$$

$$\frac{\partial \vec{T}_{\alpha(i)}}{\partial q^{\beta}} = \frac{\partial}{\partial q^{\beta}} \left(\bar{\xi}_{\alpha}\vec{e}_{\alpha} \times \vec{R}_{\alpha(i)} + \xi_{\alpha}\vec{e}_{\alpha} \right), \quad \alpha \le \beta$$

$$= \frac{\partial}{\partial q^{\beta}} \left(\bar{\xi}_{\alpha}\vec{e}_{\alpha} \right) \times \vec{R}_{\alpha(i)} + \left(\bar{\xi}_{\alpha}\vec{e}_{\alpha} \right) \times \frac{\partial \vec{R}_{\alpha(i)}}{\partial q^{\beta}} + \frac{\partial \left(\xi_{\alpha}\vec{e}_{\alpha} \right)}{\partial q^{\beta}}$$

$$\frac{\partial \vec{T}_{\alpha(i)}}{\partial q^{\beta}} = \left(\bar{\xi}_{\alpha}\vec{e}_{\alpha} \right) \times \frac{\partial \vec{R}_{\alpha(i)}}{\partial q^{\beta}} =$$

$$= \bar{\xi}_{\alpha}\vec{e}_{\alpha} \times \frac{\partial}{\partial q^{\beta}} \left(\sum_{k=1}^{\alpha-1} \left(\vec{\rho}_{kk} + \xi_k \vec{e}_k q^k \right) + \vec{\rho}_{\alpha-1} + \sum_{k=\alpha}^{i} \left(\vec{\rho}_{kk} + \xi_k \vec{e}_k q^k \right) + \vec{\rho}_i \right)$$

$$\frac{\partial \vec{T}_{\alpha(i)}}{\partial q^{\beta}} = \left(\bar{\xi}_{\alpha}\vec{e}_{\alpha} \right) \times \frac{\partial \vec{r}_{(i)}}{\partial q^{\beta}}$$

$$\frac{\partial \vec{T}_{\alpha(i)}}{\partial q^{\beta}} = \left(\bar{\xi}_{\alpha}\vec{e}_{\alpha}\right)\times \vec{T}_{\beta(i)}$$

$$\Gamma^{tr}{}_{\alpha\beta,\gamma} = \sum_{i=1}^{n} m_i \left(\bar{\xi}_{\alpha}\vec{e}_{\alpha}\times \vec{T}_{\beta(i)}\right)\cdot \vec{T}_{\gamma(i)}, \quad \alpha \le \beta \quad \Rightarrow \quad \Gamma^{tr}{}_{\alpha\beta,\gamma} = \sum_{i=1}^{n} m_i \bar{\xi}_{\alpha}\vec{e}_{\alpha}\cdot\left(\vec{T}_{\beta(i)}\times \vec{T}_{\gamma(i)}\right), \quad \alpha \le \beta$$

2x

$$\Rightarrow \quad \Gamma^{tr}{}_{\alpha\beta,\gamma} = \sum_{i=\sup(\beta,\gamma)}^{n} m_i \bar{\xi}_{\alpha}\vec{e}_{\alpha}\cdot\left(\vec{T}_{\beta(i)}\times \vec{T}_{\gamma(i)}\right), \quad \alpha \le \beta$$

$$\Gamma^{rot}{}_{\alpha\beta,\gamma} = \sum_{i=1}^{n}\int_{(Vi)} \frac{\partial^2 \vec{\rho}_i}{\partial q^{\alpha}\partial q^{\beta}}\cdot\frac{\partial \vec{\rho}_i}{\partial q^{\gamma}}\, dm_i \quad \alpha \le \beta$$

$$\frac{\partial \vec{\rho}_i}{\partial q^{\gamma}} = \vec{\Omega}_{\gamma(i)}\times\vec{\rho}_i$$

$$\frac{\partial^2 \vec{\rho}_i}{\partial q^{\alpha}\partial q^{\beta}} = \frac{\partial}{\partial q^{\beta}}\left(\frac{\partial \vec{\rho}_i}{\partial q^{\alpha}}\right) = \frac{\partial}{\partial q^{\beta}}\left(\vec{\Omega}_{\alpha(i)}\times\vec{\rho}_i\right) = \frac{\partial}{\partial q^{\beta}}\left(\vec{\Omega}_{\alpha(i)}\right)\times\vec{\rho}_i + \vec{\Omega}_{\alpha(i)}\times\frac{\partial \vec{\rho}_i}{\partial q^{\beta}}$$

$$= \vec{\Omega}_{\alpha(i)}\times\frac{\partial \vec{\rho}_i}{\partial q^{\beta}} = \vec{\Omega}_{\alpha(i)}\times\left(\vec{\Omega}_{\beta(i)}\times\vec{\rho}_i\right)$$

$$\left(\vec{\Omega}_{\alpha(i)}\times\left(\vec{\Omega}_{\beta(i)}\times\vec{\rho}_i\right)\right)\cdot\left(\vec{\Omega}_{\gamma(i)}\times\vec{\rho}_i\right)$$

A X B X C

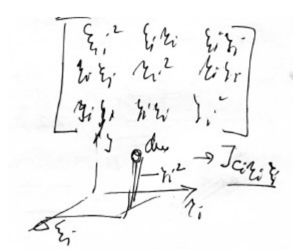
$$\vec{A}\times\left(\vec{B}\times\vec{C}\right) = \begin{vmatrix} \vec{B} & \vec{C} \\ \left(\vec{A}\cdot\vec{B}\right) & \left(\vec{A}\cdot\vec{C}\right)\end{vmatrix} = \vec{B}\left(\vec{A}\cdot\vec{C}\right) - \vec{C}\left(\vec{A}\cdot\vec{B}\right)$$

$$\vec{A}\times\left(\vec{B}\times\vec{C}\right) = \begin{vmatrix} \vec{B} & \vec{C} \\ \left(\vec{A}\cdot\vec{B}\right) & \left(\vec{A}\cdot\vec{C}\right)\end{vmatrix} = \vec{B}\left(\vec{A}\cdot\vec{C}\right) - \vec{C}\left(\vec{A}\cdot\vec{B}\right)$$

$$\left(\vec{\Omega}_{\beta(i)}\left(\vec{\Omega}_{\alpha(i)}\cdot\vec{\rho}_i\right) - \vec{\rho}_i\left(\vec{\Omega}_{\alpha(i)}\cdot\vec{\Omega}_{\beta(i)}\right)\right)\cdot\left(\vec{\Omega}_{\gamma(i)}\times\vec{\rho}_i\right)$$

$$\left(\vec{\Omega}_{\beta(i)}\left(\vec{\Omega}_{\alpha(i)}\cdot\vec{\rho}_i\right)\right)\cdot\left(\vec{\Omega}_{\gamma(i)}\times\vec{\rho}_i\right) \qquad \left(\vec{\Omega}_{\alpha(i)}\cdot\vec{\rho}_i\right)\vec{\rho}_i\cdot\left(\vec{\Omega}_{\beta(i)}\times\vec{\Omega}_{\gamma(i)}\right)$$

$$\left(\vec{\Omega}_{\beta(i)}\times\vec{\Omega}_{\gamma(i)}\right)\left\{\vec{\rho}_i\right\}\left(\vec{\rho}_i\right)\left\{\vec{\Omega}_{\alpha(i)}\right\}$$



$$\left\{\vec{\rho}_i\right\}\left(\vec{\rho}_i\right) = \begin{Bmatrix}\xi_i \\ \eta_i \\ \zeta_i\end{Bmatrix}\left(\xi_i \quad \eta_i \quad \zeta_i\right) = \begin{bmatrix}\xi_i^2 & \xi_i\eta_i & \xi_i\zeta_i \\ \eta_i\xi_i & \eta_i^2 & \eta_i\zeta_i \\ \zeta_i\xi_i & \zeta_i\eta_i & \zeta_i^2\end{bmatrix}$$

$$\left[\Pi_{Ci}\right] = \int_V \left\{\vec{\rho}_i\right\}\left(\vec{\rho}_i\right) dm_i$$ Планарни тензор инерције

$$\left[\Pi_{Ci}\right]=\begin{bmatrix}\Pi_{Ci\eta_i\zeta_i} & J_{\xi_i\eta_i} & J_{\xi_i\zeta_i}\\ J_{\xi_i\eta_i} & \Pi_{Ci\xi_i\zeta_i} & J_{\eta_i\zeta_i}\\ J_{\xi_i\zeta_i} & J_{\eta_i\zeta_i} & \Pi_{Ci\xi_i\eta_i}\end{bmatrix}$$

$$\Gamma^{rot}{}_{\alpha\beta,\gamma}=\sum_{i=1}^{n}\left(\vec{\Omega}_{\beta(i)}\times\vec{\Omega}_{\gamma(i)}\right)\left[\Pi_{Ci}\right]\left\{\vec{\Omega}_{\alpha(i)}\right\}$$

$$\Gamma^{rot}{}_{\alpha\beta,\gamma}=\sum_{i=\sup(\beta,\gamma)}^{n}\left(\vec{\Omega}_{\beta(i)}\times\vec{\Omega}_{\gamma(i)}\right)\left[\Pi_{Ci}\right]\left\{\vec{\Omega}_{\alpha(i)}\right\}$$

$$\Gamma^{rot}{}_{\alpha\beta,\gamma}=\sum_{i=\sup(\beta,\gamma)}^{n}\bar{\xi}_{\alpha}\bar{\xi}_{\beta}\bar{\xi}_{\gamma}\left(\vec{e}_{\beta}\times\vec{e}_{\gamma}\right)\left[\Pi_{Ci}\right]\left\{\vec{e}_{\alpha}\right\}$$

$$\left(\vec{e}_{\beta}\times\vec{e}_{\gamma}\right)=-\left(\vec{e}_{\gamma}\times\vec{e}_{\beta}\right)\qquad \Gamma^{rot}{}_{\alpha\beta,\gamma}=-\Gamma^{rot}{}_{\alpha\gamma,\beta}$$

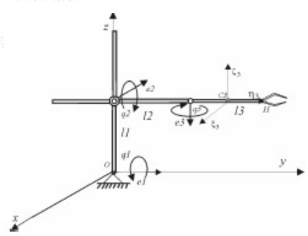
Пример израчунавања планарног тензора инерције

Primer 3. *Odrediti planarni tenzor inercije* Π_C *segmenta* $[V_3]$ *datog robotskog sistema (primer1), koji je oblika prizmatičnog štapa, mase* $m_3=3\ kg$ *i dužine* $l_3=2\ m$.

$$J_{\eta3C_3\zeta3}=\frac{\left(J_{C_3\eta}+J_{C_3\zeta}-J_{C_3\xi}\right)}{2},\quad J_{\zeta_3C_3\xi_3}=\frac{\left(J_{C_3\xi}+J_{C_3\zeta}-J_{C_3\eta}\right)}{2},$$

$$J_{\xi_3C_3\eta_3}=\frac{\left(J_{C_3\xi}+J_{C_3\eta}-J_{C_3\zeta}\right)}{2}$$

$$J_{C_3\xi}=J_{C_3\zeta}=\frac{m_3l_3^2}{12}=\frac{3\cdot2^2}{12}=1\quad kgm^2,\qquad J_{C_3\eta}=0\quad kgm^2$$



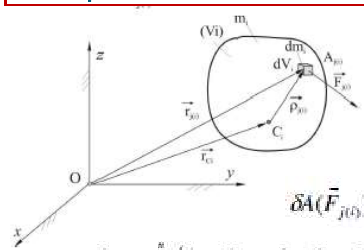
$$J_{\eta3C_3\zeta3}=\frac{\left(J_{C_3\eta}+J_{C_3\zeta}-J_{C_3\xi}\right)}{2}=0\quad kgm^2,$$

$$J_{\zeta_3C_3\xi_3}=\frac{\left(J_{C_3\xi}+J_{C_3\zeta}-J_{C_3\eta}\right)}{2}=1\quad kgm^2,$$

$$J_{\xi_3C_3\eta_3}=\frac{\left(J_{C_3\xi}+J_{C_3\eta}-J_{C_3\zeta}\right)}{2}=0\quad kgm^2,$$

$$\left[\Pi_{C3}^{(3)}\right]=\begin{bmatrix}J_{\eta3C3\zeta3} & 0 & 0\\ 0 & J_{\zeta3C3\xi3} & 0\\ 0 & 0 & J_{\xi3C3\eta3}\end{bmatrix}=\begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix}.$$

Генералисане силе у динамици крутих сила



$$A_{j(i)} \in (V_i), \qquad \vec{F}_{j(i)} \; (j = 1,2,\dots,l_i)$$

$$\delta A(\vec{F}_{j(i)}) = \vec{F}_{j(i)} \cdot \delta \vec{r}_{j(i)},$$

$$\delta A(\vec{F}_{j(i)}) = \vec{F}_{j(i)} \cdot \left(\delta \vec{r}_{C_i} + \delta\vec{\rho}_{j(i)}\right),$$

$$\delta A(\vec{F}_{j(i)}) = \vec{F}_{j(i)} \cdot \left(\sum_{\alpha=1}^{n} \vec{T}_{\alpha(i)} \delta q^{\alpha} + \sum_{\alpha=1}^{n} \vec{\Omega}_{\alpha(i)} \times \vec{\rho}_{j(i)} \right),$$

$$\delta A(\vec{F}_{j(i)}) = \sum_{\alpha=1}^{n} \left(\vec{F}_{j(i)} \cdot \vec{T}_{\alpha(i)} + \vec{M}_{Ci}(\vec{F}_{j(i)}) \cdot \vec{\Omega}_{\alpha(i)} \right), \qquad \vec{M}_{Ci}(\vec{F}_{j(i)}) = \vec{\rho}_{j(i)} \times \vec{F}_{j(i)},$$

$$\delta A^{a} = \sum_{j=1}^{l_i} \delta A(\vec{F}_{j(i)}), \quad \Rightarrow \quad \delta A_i^{a} = \sum_{\alpha=1}^{n} \left(\left(\sum_{j=1}^{l_i} \vec{F}_{j(i)} \right) \cdot \vec{T}_{\alpha(i)} + \left(\sum_{j=1}^{l_i} \vec{M}_{C_i}(\vec{F}_{j(i)}) \right) \cdot \vec{\Omega}_{\alpha(i)} \right) \cdot \delta q^{\alpha},$$

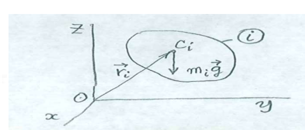
$$\delta A_i^{a} = \sum_{\alpha=1}^{n} \left(\vec{F}_{R(i)} \cdot \vec{T}_{\alpha(i)} + \vec{M}^{a}_{C_i R(i)} \cdot \vec{\Omega}_{\alpha(i)} \right) \cdot \delta q^{\alpha},$$

$$\delta A^{a} = \sum_{j=1}^{n} Q_{\alpha}^{a} \delta q^{\alpha},$$

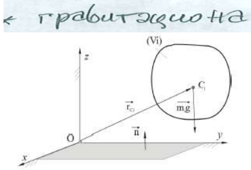
$$\delta A^{a} = \sum_{i=1}^{n} \delta A_i^{a} = \sum_{\alpha=1}^{n} \sum_{i=1}^{n} \left(\vec{F}_{R(i)} \cdot \vec{T}_{\alpha(i)} + \vec{M}^{a}_{C_i R(i)} \cdot \vec{\Omega}_{\alpha(i)} \right) \delta q^{\alpha},$$

$$Q^{a}{}_{\alpha} = \sum_{i=1}^{n} \left(\vec{T}_{\alpha(i)} \vec{F}_{R(i)} + \vec{\Omega}_{\alpha(i)} \vec{M}^{\vec{F}_{R(i)}}_{Ci(i)} \right) \qquad \alpha = 1,2 \dots n$$

$$E_p = E_p(q^1, q^2, \dots, q^n), \quad \Rightarrow \quad Q_{\alpha}^{a} = -\frac{\partial E_p}{\partial q^{\alpha}}, \; \alpha = 1,2,\dots,n.$$



← гравитациона сила



$$\vec{F}_{Ri} = -m_i g \vec{k}, \qquad \vec{M}_{C_i R} = 0$$

$$Q_{\alpha(g)} = \sum_{\alpha \le i} \vec{T}_{i(\alpha)} \cdot \vec{F}_{Ri} = -g \sum_{\alpha \le i} m_i \vec{T}_{i(\alpha)} \cdot \vec{k}$$

$$E_p = \sum_{i=1}^{n} m_i g z_i + const. = \sum_{i=1}^{n} m_i g \vec{r}_i \cdot \vec{k} + const.$$

$$Q_{\alpha(g)} = -\frac{\partial E_p}{\partial q^{\alpha}} = -\sum_{i=1}^{n} m_i g \frac{\partial \vec{r}_i}{\partial q^{\alpha}} \cdot \vec{k} = -g \sum_{\alpha \le i} m_i \vec{T}_{i(\alpha)} \cdot \vec{k}$$

$$Q_{\gamma}^{g} = -\frac{\partial E_p^{g}(\boldsymbol{q})}{\partial q^{\gamma}}, \quad \gamma = 1,2,\dots,n$$

$$Q_{\alpha(g)} = \sum_{i=\alpha}^{n=3} m_i \vec{g} \cdot \vec{T}_{\alpha(i)}$$

Primer 10 *Odrediti generalisane sile od sila teže robotskog sistema (primer 2).*
Na osnovu izvedenog izraza za generalisanu silu dobija se izraz za $Q_{3(g)}$ za treći segment:

$$Q_{3(g)} = -\sum_{i=3}^{n=3} m_i g \bar{T}_{\alpha(i)} \cdot \bar{k} = -m_3 g \left(T_{3(3)}^{(0)}\right)\left\{k^{(0)}\right\} = -m_3 g \left(\left[A_{0,3}\right]\left\{T_{3(3)}^{(3)}\right\}\right)\left\{k^{(0)}\right\} = \\ = -m_3 g \left(\left[A_{0,1}\right]\left[A_{1,2}\right]\left[I\right]\left\{T_{3(3)}^{(3)}\right\}\right)\left\{k^{(0)}\right\} \tag{5.197}$$

Generalisane sile odredjujemo u odnosu na nepokretni koordinatni sistem, što je i naznačeno u izrazima za vektore sa $(\ )^{(0)}$. U prethodnim primerima odredjen je $\left\{T_{3(3)}^{(3)}\right\}$:

$$\left\{T_{3(3)}^{(3)}\right\} = \left\{e_3^{(3)}\right\} = \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix}$$

kao i $\left[A_{0,1}\right]\left[A_{1,2}\right]$:

$$\left[A_{0,3}\right] = \left[A_{0,1}\right]\left[A_{1,2}\right]\left[A_{2,3}\right] = \left[A_{0,1}\right]\left[A_{1,2}\right]\left[I\right] = \begin{bmatrix} \cos q^1 & -\sin q^1 & 0 \\ \sin q^1 & \cos q^1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos q^2 & -\sin q^2 \\ 0 & \sin q^2 & \cos q^2 \end{bmatrix} =$$

$$= \begin{bmatrix} \cos q^1 & -\sin q^1 \cos q^2 & \sin q^1 \sin q^2 \\ \sin q^1 & \cos q^1 \cos q^2 & -\sin q^2 \cos q^1 \\ 0 & \sin q^2 & \cos q^2 \end{bmatrix}$$

$$Q_{3(g)} = -5 \cdot 9{,}81 \left(\begin{bmatrix} \cos q^1 & -\sin q^1 \cos q^2 & \sin q^1 \sin q^2 \\ \sin q^1 & \cos q^1 \cos q^2 & -\sin q^2 \cos q^1 \\ 0 & \sin q^2 & \cos q^2 \end{bmatrix}\begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix}\right)\begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix} =$$

$$= -49{,}05\left(-\sin q^1 \cos q^2 \quad \cos q^1 \cos q^2 \quad \sin q^2\right)\begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix} = -49{,}05 \sin q^2$$

Генералисана сила од сила у опрузи

$$E_p^c = \frac{1}{2} c \left(\overline{F_k F_j} - l_0\right)^2,$$

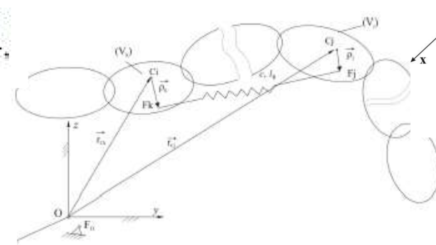
$$Q_\alpha^c = -c\left(\overline{F_k F_j} - l_0\right)\frac{\partial \overline{F_k F_j}}{\partial q^\alpha},$$

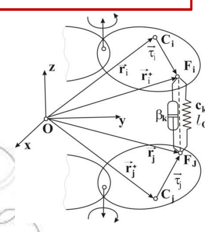
$$\overline{F_k F_j} = \sqrt{\overrightarrow{F_k F_j} \cdot \overrightarrow{F_k F_j}},$$

$$\frac{\partial\left(\overline{F_k F_j}\right)}{\partial q^\alpha} = \frac{\overrightarrow{F_k F_j}}{\overline{F_k F_j}} \cdot \frac{\partial\left(\overrightarrow{F_k F_j}\right)}{\partial q^\alpha},$$

$$\overrightarrow{F_k F_j} = \bar{r}_{C_j} + \bar{\rho}_j - \bar{r}_{C_k} - \bar{\rho}_k,$$

$$\frac{\partial\left(\overline{F_k F_j}\right)}{\partial q^\alpha} = \frac{\overrightarrow{F_k F_j}}{\overline{F_k F_j}} \cdot \left(\bar{T}_{\alpha(j)} - \bar{T}_{\alpha(k)} + \bar{\Omega}_{\alpha(j)} \times \bar{\rho}_j - \bar{\Omega}_{\alpha(k)} \times \bar{\rho}_k\right),$$





$$\overline{F_k F_j} = \left|\bar{r}_{C_j} + \bar{\rho}_j - \bar{r}_{C_k} - \bar{\rho}_k\right|,$$

$$Q_\alpha^c = -c\left(\left|\bar{r}_{C_j} - \bar{r}_{C_k} + \bar{\rho}_j - \bar{\rho}_k\right| - l_0\right)\frac{\bar{r}_{C_j} - \bar{r}_{C_k} + \bar{\rho}_j - \bar{\rho}_k}{\left|\bar{r}_{C_j} - \bar{r}_{C_k} + \bar{\rho}_j - \bar{\rho}_k\right|} \cdot \left(\bar{T}_{\alpha(j)} - \bar{T}_{\alpha(k)} + \bar{\Omega}_{\alpha(j)} \times \bar{\rho}_j - \bar{\Omega}_{\alpha(k)} \times \bar{\rho}_k\right).$$

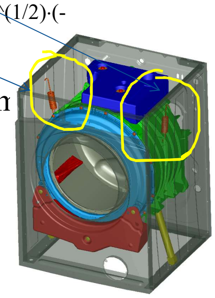
Generalized spring forces

$$E_p^c = \frac{1}{2}c\left(l_{A_1B_1} - l_o\right)^2 + \frac{1}{2}c\left(l_{A_2B_2} - l_o\right)^2$$

$$Q_\gamma^c = -\frac{\partial E_p^c}{\partial q^\gamma} = -\sum_{k=1}^{m} c_k\left(l_{ij} - l_{0k}\right)\frac{\partial l_{ij}}{\partial q^\gamma}, \quad \alpha = 1,2,\dots,n$$

$Q_{\gamma(1)}^c \rightarrow$

Qc(1)= -1/2·c· ((((xb1-q1- ua1·cos(q3) +va1·sin(q3)) ^2 +(yb1-q2-ua1·sin(q3)-va1·cos(q3))^2)^(1/2)-lo)/((xb1-q1-ua1·cos(q3)+va1·sin(q3))^2+(yb1-q2-ua1·sin(q3)-va1·cos (q3))^2)^(1/2)·(-2·xb1+ 2·q1+2·ua1·cos(q3)-2·va1 ·sin(q3))+(((xb2-q1-ua2·cos(q3)+va2·sin(q3))^2+(yb2-q2-ua2· sin (q3)-va2·cos(q3))^2)^(1/2)-lo)/((xb2-q1-ua2·cos (q3) + va2·sin(q3))^2+(yb2-q2-ua2·sin(q3)-va2· cos(q3)) ^2)^(1/2)·(-2·xb2+2·q1+2·ua2·cos(q3)-2·va2 ·sin(q3))),

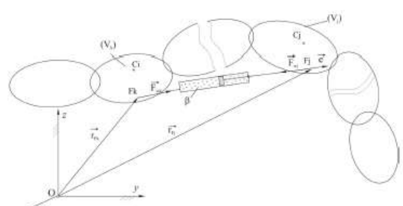


using Symbolic toolbox-a -differential equations of n

Q1=(m+me)·q1dd-m·(sin(q3)·ut+cos(q3)·vt)·q3dd-me·e· sin(q4)·q4dd-m·(cos(q3)·ut-sin(q3)·vt)·q3d^2-me·e·cos(q4) ·q4d^2
Q2=(m+me)·q2dd+m·(cos(q3)·ut-sin(q3)·vt)·q3dd+me·e·cos(q4)·q4dd+m·(-sin(q3)·ut-s(q3)·vt)·q3d^2-me·e·sin(q4) ·q4d^2
Q3=-m·(sin(q3)·ut+cos(q3)·vt)·q1dd+m·(cos(q3)·ut-sin(q3)·vt)·q2dd+(m·ut^2+m·vt^2+jt) ·q3dd
Q4=-me·e·sin(q4)·q1dd+me·e·cos(q4)·q2dd+(jte+me·e^2) ·q4dd

- Генералисана сила од сила пригушивача (вискозно трење)

$$\vec{F}_{wj} = -\vec{F}_{wk} = -\beta\vec{v}_{rk},$$



$$\vec{F}_w = -\bar{\beta}\vec{v}_r \rightarrow \delta A_{(w)} = \vec{F}_w\cdot\delta\vec{l} = -\bar{\beta}\frac{d\vec{l}}{dt}\cdot\delta\vec{l} = -\bar{\beta}\frac{dl}{dt}\delta l$$

$$\frac{dl}{dt} = \sum_{\beta=1}^{n}\frac{\partial l}{\partial q^\beta}\dot{q}^\beta, \quad \delta l = \sum_{\alpha=1}^{m}\frac{\partial l}{\partial q^\alpha}\delta q^\alpha \rightarrow \delta A_{(w)} = -\bar{\beta}\sum_{\beta=1}^{m}\frac{\partial l}{\partial q^\beta}\dot{q}^\beta\sum_{\alpha=1}^{n}\frac{\partial l}{\partial q^\alpha}\delta q^\alpha$$

$$\delta A_{(w)} = -\sum_{\alpha=1}^{m}\bar{\beta}\frac{\partial l}{\partial q^\alpha}\left(\sum_{\beta=1}^{m}\frac{\partial l}{\partial q^\beta}\dot{q}^\beta\right)\delta q^\alpha = \sum_{\alpha=1}^{m}Q_{\alpha(w)}\delta q^\alpha$$

$$Q_{\alpha(w)} = -\bar{\beta}\frac{\partial l}{\partial q^\alpha}\left(\sum_{\beta=1}^{m}\frac{\partial l}{\partial q^\beta}\dot{q}^\beta\right)$$

$$\Phi = \frac{1}{2}\bar{\beta}\, V_r^2 = \frac{1}{2}\bar{\beta}\left(\frac{dl}{dt}\right)^2 = \frac{1}{2}\bar{\beta}\sum_{\alpha=1}^{n}\sum_{\beta=1}^{n}\frac{\partial l}{\partial q^\alpha}\frac{\partial l}{\partial q^\beta}\dot{q}^\alpha\dot{q}^\beta \Rightarrow Q_{\alpha(w)} = -\frac{\partial\Phi}{\partial\dot{q}^\alpha}$$

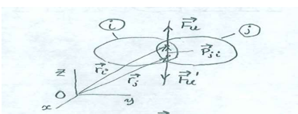
$$Q_{\alpha(w)} = -\bar{\beta}\frac{\partial l}{\partial q^\alpha}\sum_{\beta=1}^{n}\frac{\partial l}{\partial q^\beta}\dot{q}^\beta$$

$$Q_\alpha^\beta = -\beta\,\frac{\partial l_k}{\partial q^\alpha}\sum_{\beta=1}^{n}\frac{\partial l_k}{\partial q^\beta}\dot{q}^\beta$$

$$Q_\alpha^{\,w} = -\beta\sum_{\beta=1}^{n}\frac{\overline{F_kF_j}}{\overline{F_kF_j}}\cdot\frac{\partial\left(\overline{F_kF_j}\right)}{\partial q^\beta}\frac{\overline{F_kF_j}}{\overline{F_kF_j}}\cdot\frac{\partial\left(\overline{F_kF_j}\right)}{\partial q^\alpha}\dot{q}^\beta,$$

$$Q_\alpha^w = -\beta\sum_{\beta=1}^{n}\frac{\vec{r}_{C_j}-\vec{r}_{C_k}+\vec{\rho}_j-\vec{\rho}_k}{\left|\vec{r}_{C_j}-\vec{r}_{C_k}+\vec{\rho}_j-\vec{\rho}_k\right|^2}\cdot\left(\vec{T}_{\beta(j)}-\vec{T}_{\beta(k)}+\vec{\Omega}_{\beta(j)}\times\vec{\rho}_j-\vec{\Omega}_{\beta(k)}\times\vec{\rho}_k\right)*$$
$$*\left(\vec{r}_{C_j}-\vec{r}_{C_k}+\vec{\rho}_j-\vec{\rho}_k\right)\cdot\left(\vec{T}_{\alpha(j)}-\vec{T}_{\alpha(k)}+\vec{\Omega}_{\alpha(j)}\times\vec{\rho}_j-\vec{\Omega}_{\alpha(k)}\times\vec{\rho}_k\right)\dot{q}^\beta,$$

- Генералисана сила од система погонских сила

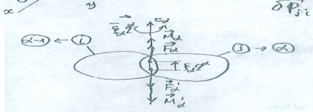


$\vec{F}_u, \vec{F}_u'$ – погонске силе (унутрашње силе система)

$\delta A_{(u)} = \vec{F}_u\cdot\delta\vec{r}_i + \vec{F}_u'\cdot\delta\vec{r}_j = \vec{F}_u(\delta\vec{r}_i-\delta\vec{r}_j) = \vec{F}_u\cdot\delta\vec{\rho}_{ji}$

$\delta\vec{\rho}_{ji} = \xi_j\vec{e}_j\delta q^j$ – за транслаторни зглоб

$\delta\vec{\rho}_{ji} = \bar{\xi}_j\vec{e}_j\delta q^j$ – за ротациони зглоб

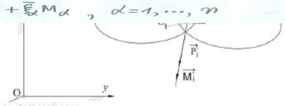


вектори погонских сила и вектори момената погонских спрегова су колинеарни са осом кинематичког пара па следи:

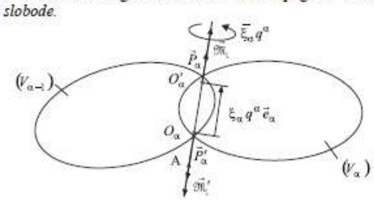
$$\delta A_{(p)} = \sum_{\alpha=1}^{n}(\xi_\alpha F_\alpha\delta q^\alpha + \bar{\xi}_\alpha M_\alpha\delta q^\alpha) = \sum_{\alpha=1}^{n}\delta q^\alpha(\xi_\alpha F_\alpha + \bar{\xi}_\alpha M_\alpha)$$

$$\delta A_{(p)} = \sum_{\alpha=1}^{n}Q_{\alpha(p)}\delta q^\alpha \Rightarrow Q_{\alpha(p)} = \xi_\alpha F_\alpha + \bar{\xi}_\alpha M_\alpha,\quad \alpha=1,\dots,n$$

$$\delta q^1 = \delta q^2 = \ldots = \delta q^{\alpha-1}, \delta q^\alpha \neq 0, \delta q^{\alpha+1} = \delta q^{\alpha+2} = \ldots = \delta q^n = 0.$$



Primer 15 *Izvesti izraz za generalisanu silu od sila pogona robotskog sistema sa n stepeni slobode.*



$$\delta A(\bar{P}_\alpha, \bar{P}'_\alpha) = \bar{P}_\alpha \delta(AO'_\alpha \bar{e}_\alpha) + \bar{P}'_\alpha \delta(AO_\alpha \bar{e}_\alpha) = \bar{P}_\alpha \delta(AO_\alpha \bar{e}_\alpha + \xi_\alpha q^\alpha \bar{e}_\alpha) + \bar{P}'_\alpha \delta(AO_\alpha \bar{e}_\alpha) =$$
$$= \bar{P}_\alpha \bar{e}_\alpha \delta(AO_\alpha + \xi_\alpha q^\alpha) + \bar{P}'_\alpha \bar{e}_\alpha \delta(AO_\alpha) = \bar{P}_\alpha \bar{e}_\alpha \delta q^\alpha$$

$$\overline{AO}_\alpha = const \Rightarrow \delta(\overline{AO}_\alpha) = 0, \quad A \in osi \;\; rotacije, translacije$$

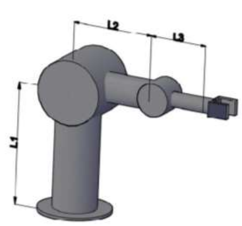
$$\delta A(\bar{P}_\alpha, \bar{P}'_\alpha) = Q_{\alpha(P)} \delta q^\alpha = \bar{P}_\alpha \bar{e}_\alpha \delta q^\alpha \Rightarrow Q_{\alpha(P)} = \bar{P}_\alpha \bar{e}_\alpha$$

$$\delta A(\bar{M}_\alpha, \bar{M}'_\alpha) = \bar{M}_\alpha \delta(\bar{\xi}_\alpha q^\alpha \bar{e}_\alpha) + \bar{M}'_\alpha \delta(0) = \bar{M}_\alpha \bar{e}_\alpha \delta q^\alpha$$

$$\delta A(\bar{M}_\alpha, \bar{M}'_\alpha) = Q_{\alpha(M)} \delta q^\alpha = \bar{M}_\alpha \bar{e}_\alpha \delta q^\alpha \Rightarrow Q_{\alpha(M)} = \bar{M}_\alpha \bar{e}_\alpha$$

$$Q_{\alpha(pog)} = Q_{\alpha(P)} + Q_{\alpha(M)} = \xi_\alpha \bar{P}_\alpha \bar{e}_\alpha + \bar{\xi}_\alpha \bar{M}_\alpha \bar{e}_\alpha$$

ПРИМЕР РОБОТА СА 3 СС (R-R-R)



i	l_i [m]	d_i [m]	m_i [kg]	$J_{xi}=J_{yi}$ [m⁴]	J_{zi} [m⁴]
1	0,8	0,3	15	0.8843	0.1687
2	0,4	0,2	8	0.1267	0.04
3	0,3	0,1	5	0.0406	0.0062

$$\vec{e}_1^{(1)} = (001)^T, \vec{e}_2^{(2)} = \vec{e}_3^{(3)} = (100)^T$$

$$\vec{\rho}_{11}^{(1)} = \begin{Bmatrix} 0 \\ 0 \\ 0.8 \end{Bmatrix} \; \vec{\rho}_{22}^{(2)} = \begin{Bmatrix} 0 \\ 0.4 \\ 0 \end{Bmatrix} \; \vec{\rho}_{33}^{(3)} = \begin{Bmatrix} 0 \\ 0.3 \\ 0 \end{Bmatrix}$$

$$\vec{\rho}_1^{(1)} = \begin{Bmatrix} 0 \\ 0 \\ -0.4 \end{Bmatrix} \; \vec{\rho}_{22}^{(2)} = \begin{Bmatrix} 0 \\ -0.2 \\ 0 \end{Bmatrix} \; \vec{\rho}_{33}^{(3)} = \begin{Bmatrix} 0 \\ -0.15 \\ 0 \end{Bmatrix}$$

Диф. једначине кретања - simbolicki oblik primenom MATLAB® Symbolic Toolbox

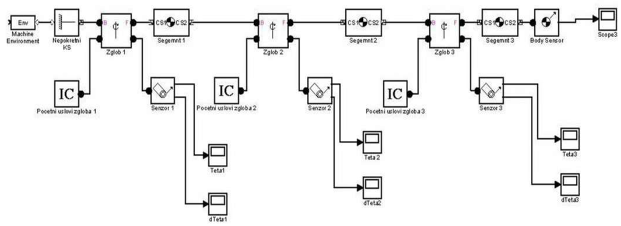
Qp1=1789/10000+320000·cos(q1)^2·cos(q2)^2+320000·sin(q1)^2·cos(q2)^2+12500·cos(q1)^2·(8·cos(q2)+3·cos(q2) ·cos(q3)-
-3·sin(q2) ·sin(q3))^2+12500·sin(q1)^2· (8·cos(q2)+3·cos(q2) ·cos(q3)-3·sin(q2) ·sin(q3))^2) ·d2q1+2· (-320000·cos(q1)^2 ·cos(q2)·sin(q2)-320000·sin(q1)^2·cos(q2) ·sin(q2)+ 12500·cos(q1)^2· (8·cos(q2)+3·cos(q2) ·cos(q3)-3·sin(q2) ·sin(q3)) · (-8*sin(q2)-3·sin(q2) ·cos(q3)-3·cos(q2) ·sin(q3))+12500·sin(q1)^2· (8·cos(q2)+3·cos(q2) ·cos(q3)-3·sin(q2) ·sin(q3)) · (-8·sin(q2)-3·sin(q2) ·cos(q3)-3·cos(q2) ·sin(q3))) ·dq1·dq2+2·(12500·cos(q1)^2· (8·cos(q2)+3·cos(q2) ·cos(q3)- 3·sin(q2) ·sin(q3)) · (-3·cos(q2) ·sin(q3)-3·sin(q2) ·cos(q3))+ 12500·sin(q1)^2· (8·cos(q2)+ +3·cos(q2)·cos(q3)-3·sin(q2) ·sin(q3))·(-3·cos(q2)·sin(q3)-3·sin(q2)·cos(q3))) ·dq1·dq3

Qp2=(320000·sin(q1)^2·sin(q2)^2+320000·cos(q1)^2+320000·cos(q2)^2+1673/10000+...

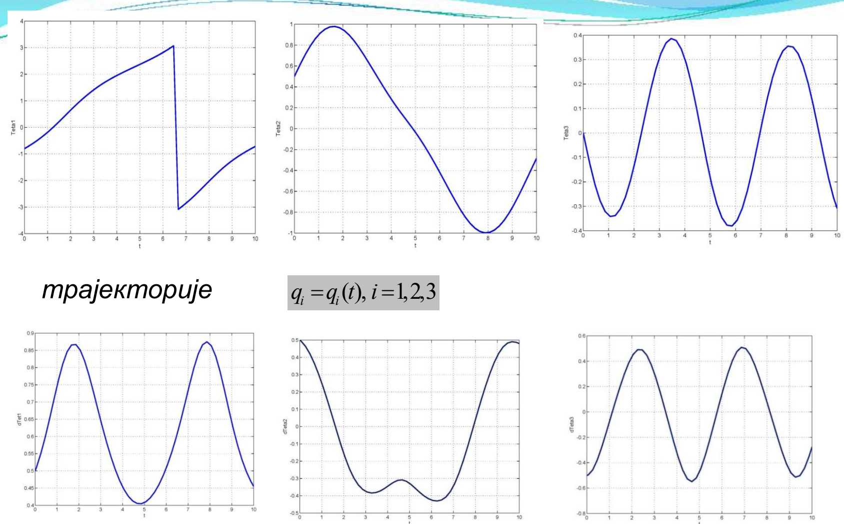
Qp3 =12500·sin(q1)^2· (-8·sin(q2)-3·sin(q2) ·cos(q3)-3·cos(q2) ·sin(q3)) · (-3·cos(q2) ·sin(q3)-...

Tabela 2: *Početni uslovi sistema*

i	$q_i(0)$	$dq_i / dt\|_{t_0=0}$
1	-0.8	0.5
2	0.5	0.5
3	0	-0.5



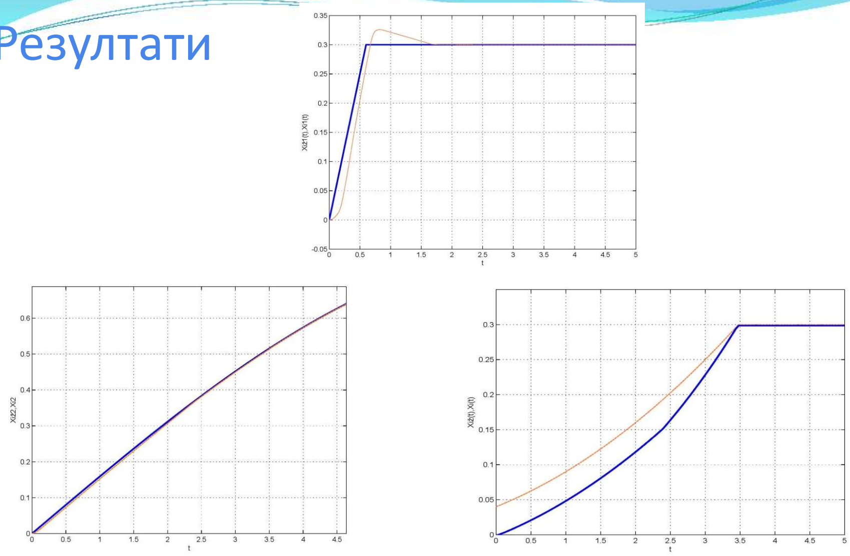
SIMULINK model neupravljačkog robotskog sistema sa tri stepena slobode



трајекторије $q_i = q_i(t),\ i = 1,2,3$

Ген. брзине $\dot{q}_i = \dot{q}_i(t),\ i = 1,2,3$

Резултати



PRIMER MODELIRANJA_ VES MASINE:Drum-type washing machine

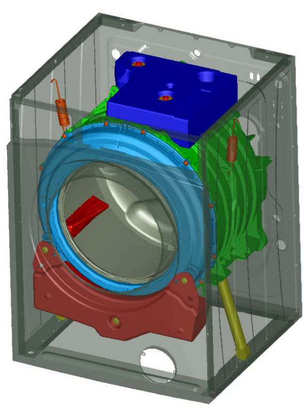


Figure 1a:
Horizontal axis washing machine

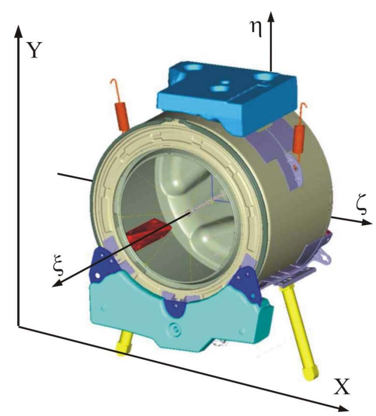
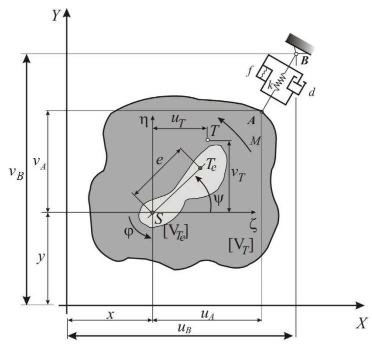
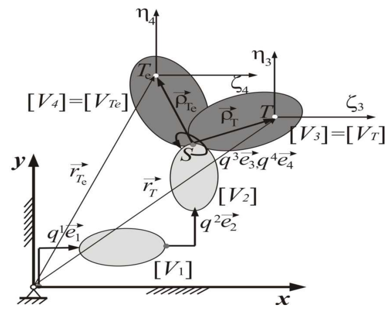


Figure 1b:Washing tub assembly
of washing machine with suspension

2D model of washing machine



. 2D model - classical approach



2D model - robotically approach

4 (DOFs) $q^1 = x, q^2 = y, q^3 = \varphi, q^4 = \psi,$

position [VT] $\overrightarrow{r_T} = q^1\vec{e}_1 + q^2\vec{e}_2 + \vec{\rho}_{33T} + \vec{\rho}_{3T}$

position [VTe] $\overrightarrow{r_{Te}} = q^1\vec{e}_1 + q^2\vec{e}_2 + \vec{\rho}_{33Te} + \vec{\rho}_{3Te}$

kinetic energy WM $E_k = \frac{1}{2}m\left(\dot{x}_T^2 + \dot{y}_T^2\right) + \frac{1}{2}J_T\left(\dot{q}^3\right)^2 + \frac{1}{2}m\left(\dot{x}_{Te}^2 + \dot{y}_{Te}^2\right) + \frac{1}{2}J_{Te}\left(\dot{q}^4\right)^2$

Generalized spring forces

$$E_p^c = \frac{1}{2}c\left(l_{A_1B_1} - l_o\right)^2 + \frac{1}{2}c\left(l_{A_2B_2} - l_o\right)^2 \qquad Q_\gamma^c = -\frac{\partial E_p^c}{\partial q^\gamma} = -\sum_{k=1}^{m} c_k\left(l_{ij} - l_{0k}\right)\frac{\partial l_{ij}}{\partial q^\gamma}, \quad \alpha = 1,2,\dots,n$$

$Q_{\gamma(1)}^c \rightarrow$ Qc(1)= -1/2·c· ((((xb1-q1- ua1·cos(q3) +va1·sin(q3)) ^2 +(yb1-q2-ua1·sin(q3)-va1·cos(q3))^2)^(1/2)-lo)/((xb1-q1-ua1·cos(q3)+va1·sin(q3))^2+(yb1-q2-ua1·sin(q3)-va1·cos (q3))^2)^(1/2)·(-2·xb1+ 2·q1+2·ua1·cos(q3)-2·va1 ·sin(q3))+(((xb2-q1-ua2·cos(q3)+va2·sin(q3))^2+(yb2-q2-ua2· sin (q3)-va2·cos(q3))^2)^(1/2)-lo)/((xb2-q1-ua2·cos (q3) + va2·sin(q3))^2+(yb2-q2-ua2·sin(q3)-va2· cos(q3)) ^2)^(1/2)·(-2·xb2+2·q1+2·ua2·cos(q3)-2·va2 ·sin(q3))),

using Symbolic toolbox-a differential equations of motions WM

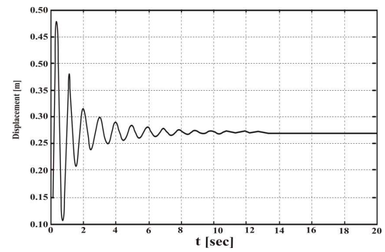
Q1=(m+me)·q1dd-m·(sin(q3)·ut+cos(q3)·vt)·q3dd-me·e· sin(q4)·q4dd-m·(cos(q3)·ut-sin(q3)·vt)·q3d^2-me·e·cos(q4) ·q4d^2
Q2=(m+me)·q2dd+m·(cos(q3)·ut-sin(q3)·vt)·q3dd+me·e·cos(q4)·q4dd+m·(-sin(q3)·ut-s(q3)·vt)·q3d^2-me·e·sin(q4) ·q4d^2
Q3=-m·(sin(q3)·ut+cos(q3)·vt)·q1dd+m·(cos(q3)·ut-sin(q3) ·vt)·q2dd+(m·ut^2+m·vt^2+jt) ·q3dd
Q4=-me·e·sin(q4)·q1dd+me·e·cos(q4)·q2dd+(jte+me·e^2) ·q4dd

Simulation results -tub assembly type PS-O3S

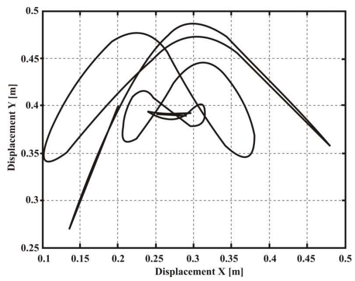
x [m]	y [m]	φ_{st} [rad]	ψ_{st} [rad]
0,2	0,4	-0,01	$-\pi/2$

–static equilibrium state - impulse response

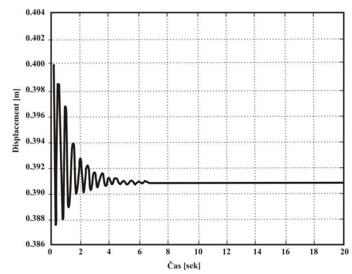
x_{st} [m]	y_{st} [m]	φ_{st} [rad]	ψ_{st} [rad]
0,2715	0,3908	-0,0042	$-\pi/2$



Displacement in X axis



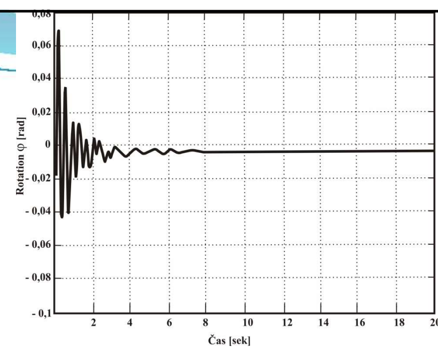
Displacement in *XY* plane



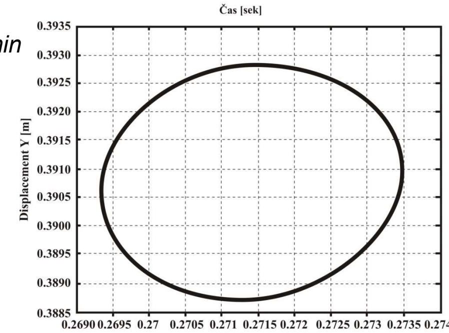
Displacement in y axis

Angular change

$$\varphi = \varphi(t)$$



for the angular speed ω = *1000 rev/min* = 104,72 rad/s ,(frequency *16,67 Hz*



Displacement in XY plane for steady state response