1 Introduction

1.1 CONCEPT AND MAIN TASKS

Flight dynamics is the engineering science about the motion (or flight) of the object in the atmosphere. This book includes chapters related to the classical projectiles' flight dynamics, such as artillery shells, uncontrolled rocket, and missile flight dynamics. To calculate the object's trajectory and angular motion about the centre of gravity, we need aerodynamic forces and moments obtained from the other engineering science – Aerodynamics.

The term "External Ballistics" is often used for unguided projectiles and rockets' flight dynamics. On the other hand, "Flight Dynamics" is the term that is more often used in aviation or related to missiles. Keeping in mind the role of aerodynamics in modern external ballistics, recently we have also had a new name, Aeroballistics.

The term projectile is a name used generally and can refer to artillery ammunition, a bomb, a guided or unguided rocket, etc.

Flight dynamics of projectiles is a part of applied mechanics and primarily relies on the dynamics of the motion of a rigid body. Still, it is also very closely connected with aerodynamics and fundamental sciences such as mathematics, thermodynamics and fluid mechanics. The block diagram of the primary subjects of the flight dynamics with its relationships with the other science and engineering disciplines is given in Figure 1-1.

Flight dynamics solves two main tasks:

1. **The primary task of flight dynamics - direct**

   Determining the trajectory characteristics for a projectile of known features, from one particular moment to a fall or other given point with a given accuracy.

2. **The inverse task of flight dynamics - design**

   Determining the design characteristics of the projectile that will enable the required flight characteristics (range, dispersion, etc.) under certain flight conditions.

Other tasks that flight dynamics deals with are:

1. **Projectile stability**

   The ability of a projectile to maintain a sufficiently small angle between the projectile axis and the velocity vector during flight is called stability. Flight dynamics studies the conditions under which the stability of the projectile is achieved, the laws of motion around the centre of gravity, and the influence that the rotation around the centre of gravity has on the movement of the centre of gravity.

2. **Determining differential coefficients (corrections)**

   Since the parameters that affect the movement of projectiles change, it is necessary to determine how their changes affect the trajectory elements.

3. **Experimental external ballistics**

   Methods for measuring quantities that characterise the trajectory and flight conditions of a projectile.

4. **Firing tables (unguided projectiles)**

   Determining the firing elements, i.e. the position and orientation of the launcher, so that the fired projectile hits the target at a certain point in space.

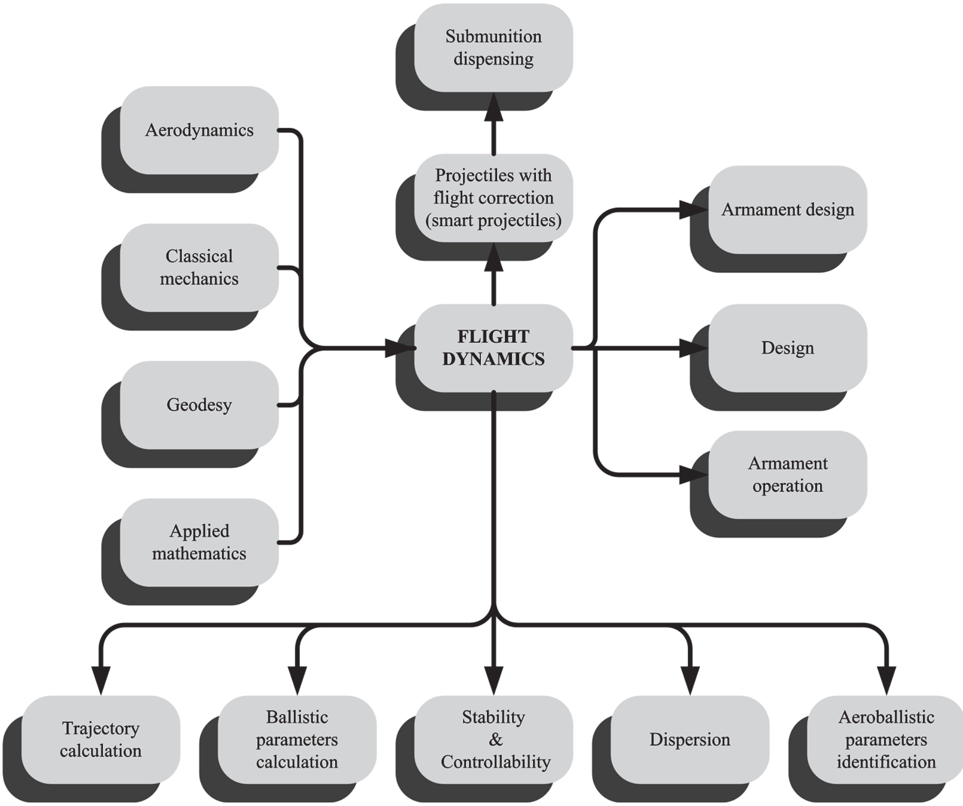


FIGURE 1-1 SUBJECTS OF FLIGHT DYNAMICS OF PROJECTILES AND ITS RELATIONSHIPS WITH THE OTHER BRANCHES OF ENGINEERING SCIENCE

In order to solve the primary task of flight dynamics, it is necessary to define the following parameters:

- Aerodynamic characteristics (coefficients of aerodynamic forces and moments as a function of Mach number)
- Characteristics of the propulsion system (thrust and mass flow as a function of time)

- Characteristics of the guidance system, if it is a guided missile (guidance and autopilot algorithm)
- Characteristics of the control system (type of control, transfer functions of actuators)
- Inertial characteristics (mass and position of the centre of gravity as a function of time)
- Atmospheric properties (standard atmosphere, deviation from standard atmosphere, wind data)
- Earth shape and gravitational field model
- Initial parameters (initial position, initial angles, etc.)

If all these parameters are known, given as an input data set, differential equations of motion can be integrated to determine the projectile's behaviour during the flight, ultimately providing the trajectory.

It is possible to define several different models of projectile motion:

- Two degrees of freedom - 2-DOF (translation in the vertical plane, without changing the azimuth angle), material point motion model (Euler's model)
- Three degrees of freedom - 3-DOF (translation in x, y, z-direction) material point motion model, or translation in x and z direction and rotation about longitudinal (x) or transversal (y) axis as 3rd degree of freedom.
- Six degrees of freedom - 6-DOF (translation in x, y, z-direction), rigid body (with three rotations around the centre of gravity.)
- An elastic model with several degrees of freedom considers elastic deformations of the projectile body during flight.

The first model, which includes the gravity and the drag force, is a practical and accurate approximation for projectiles with a small angle of attack. The last one is of interest only for large projectiles and complex structures, so that 3-DOF and 6-DOF models have found the most comprehensive application. In the 3-DOF model, the aerodynamic model of the projectile is simplified. The 3-DOF model is usually used in the preliminary optimisation of input parameters (inverse flight dynamics task) due to the speed at which the solution is reached. The 6-DOF model is used in the next phase to test the behaviour of the actual object.

The inverse task is solved by successive use of 3-DOF and 6-DOF models, with manual or automatic adjustment of the input data to achieve optimal conditions for a given trajectory. Ballistic parameters and optimal design characteristics are evaluated to meet often contradictory requirements during design.

1.2 BRIEF HISTORY

During the Second World War and after, the external ballistics had rapid progress as a result of three particular developments:

1. Fully operational supersonic wind tunnels,
2. Free flight spark photography ranges, and
3. High-speed computers.

Supersonic wind tunnels permitted direct measurements of the static aerodynamic forces and moments acting on high-speed projectiles. The effects of parametric variations in projectile shape were investigated with the new wind tunnel facilities.

Spark photography ranges permitted the precise, interference-free measurement of the drag, spin, and angular motion in flight. Using complete ballistic theory, all significant aerodynamic forces and moments acting on the projectile could be determined from a set of firings through the spark photography range.

The high-speed computer was used to reduce and analyse both wind tunnel and spark photography range data. In addition, the electronic computer rapidly solved both the differential equations of the projectile's trajectory and the partial differential equations of the flow field around the projectile.

C. H. Murphy revised and improved the stability theory and the data reduction techniques for free-flight spark photography ranges (1953-54). J. D. Nicolaides introduced the tricyclic theory of projectile's angular motion (1953), which accounted for the effect of small configurational or mass asymmetries on the epicyclic angular motion of projectiles. By 1960, Murphy advanced the quasi-linear theory, which permitted the determination of nonlinear aerodynamic forces and moments from free flight spark photography range tests. Comparison of the nonlinear spark range measurements with those obtained from the large angle of attack testing in wind tunnels showed excellent agreement. The quasi-linear technique was augmented in 1970 by Chapman and Kirk's work to identify the nonlinear aerodynamic coefficients by using six-degrees-of-freedom differential equations and the theory of the sensitivity.

The calculation of a projectile's trajectory was the classical exterior ballistics' problem. The entire modern science of external ballistics has evolved because of the continuous need to improve trajectory calculations' accuracy. The method of trajectory calculation based on the reference standard drag shapes and the application of a "form factor" relating to the new shell to one of the reference standards was disappeared after the Second World War due to the development of modern computers.

By 1960, the size and speed of digital computers had advanced to a level that permitted the first practical modern six-degrees-of-freedom (6-DOF) trajectory calculations. The 6-DOF trajectory calculation numerically solves 12 first-order differential equations to yield a complete description of the projectile's position, velocity, time, and angular motion (pitching, yawing, and spinning) from the muzzle of the gun to the target.

Modern 6-DOF trajectory calculations, based on the measured physical properties of the atmosphere, complete aerodynamic forces and moments, and all required initial conditions, have shown excellent agreement with the results of instrumented field firings. There are many situations where the older methods are entirely sufficient for all practical purposes. For example, this is true in the case of trajectory calculation of ground-launched small arms that have flat-trajectory with a small angle of attack. In this case, we can apply the point-mass trajectory (2-DOF) model, which includes only drag and gravity force. Artillery and mortar fire at very high gun elevation angles require 6-DOF calculation because of the attack angle's significant values near the trajectory's summit. More straightforward methods, such as point-mass trajectory calculation, are sufficiently accurate for artillery shells fired at lower departure angles.

1.3 ELEMENTS OF A TRAJECTORY

The missile trajectory is the geometric position of points that describes the centre of mass of the projectile during the flight. In the general case, the trajectory is a space curve. Most commonly, it is observed in a rectangular, right coordinate frame $Oxyz$, which is defined as follows:

O - Origin corresponding to launcher position;

x - Axis is the intersection of a vertical plane through the launcher axis and the horizontal plane through the origin; the horizontal component of velocity determines the positive direction;

y - Axis is vertical with a positive direction upward;

z - Axis is positive clockwise.

The vertical plane Oxy is ***the firing plane***. The direction of the horizontal axis Ox determines ***the firing direction***.

The velocity at the launcher point has a value V_0 and is called ***the initial velocity***. The initial velocity relative to the horizontal plane makes the initial angle γ_0. The angle between the major axis of symmetry of the projectile and the horizontal plane at the launcher point is called the elevation El. Usually, it takes $\gamma_0 = El$.

For a typical ballistic trajectory that begins and ends in the same horizontal plane, in addition to the starting point (O), the characteristic points are ***apogee*** (A) and endpoint (T)

We differentiate between two phases of trajectory for rockets: ***active phase*** during rocket motor burning and ***passive phase*** after the termination of the rocket motor.

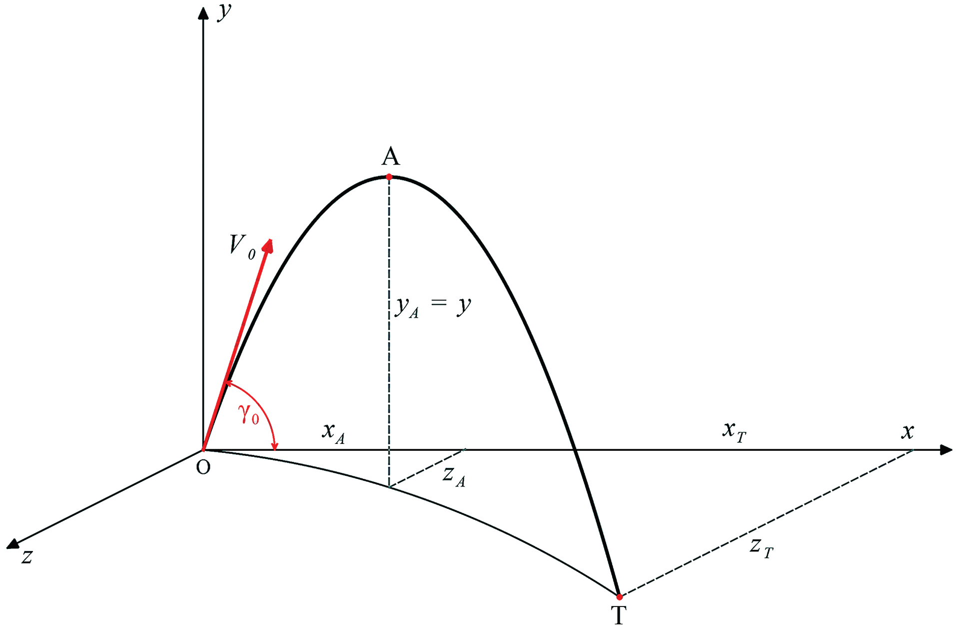


FIGURE 1-2 BASIC TRAJECTORY ELEMENTS

Essential elements of a trajectory in the firing plane are shown in Figure 1-3. The range from the origin (muzzle of a gun) to the target is called ***the map range***. The base of the trajectory is defined as a level in a plane with a firing point. The weapon is levelled in the direction depicted in Figure 1-3, as well as in the plane out the paper. In order to aim at the target, the gunner uses ***the line of sight*** and ***the angle of sight***. The line of departure is not collinear with the elevation of the weapon (i.e., the line where the bore is pointed). The difference between ***the elevation angle*** and the angle of the line of departure is due to the dynamics of the projectile and gun as well as aerodynamic effects. The projectile's out-of-plane angular position at the muzzle exit is known as lateral or ***azimuthal jump***. This will be combined with a vertical jump to give a resultant jump vector. The angle of lift and line of fall are defined for the level point. At the target, we define the line and ***angle of lift or the angle of impact***.

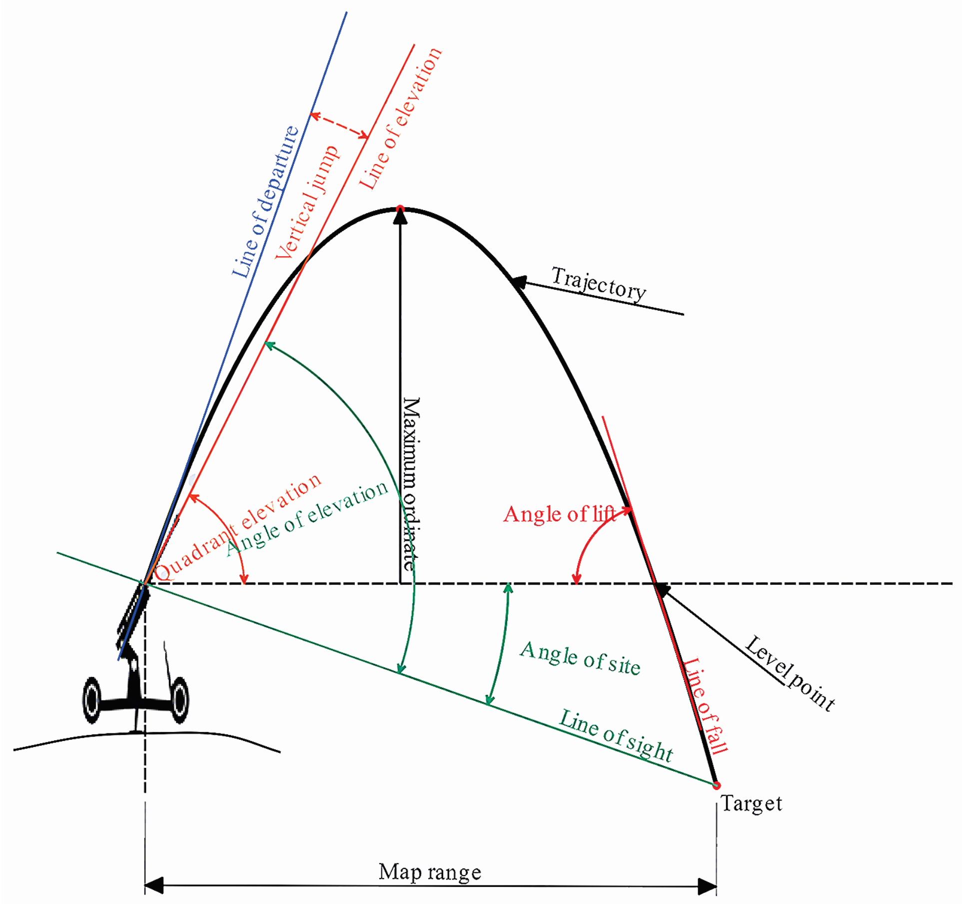


FIGURE 1-3 ELEMENTS OF A TRAJECTORY IN THE FIRING PLANE

The main reasons for the dispersion of projectiles at the target are shown in Figure 1-4. If we need a point accuracy (or single-shot hit), it is required to design a guided weapon.

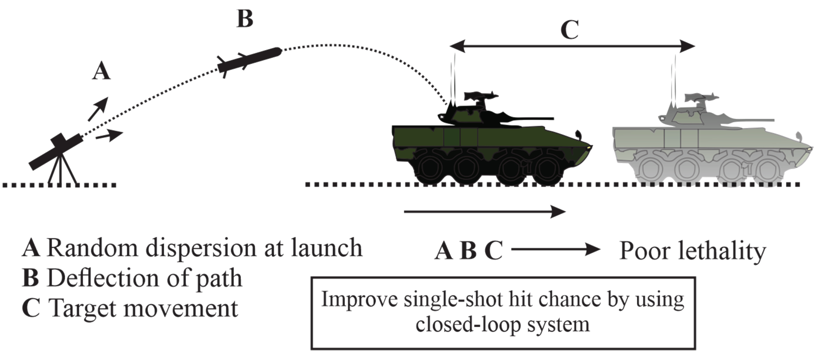


FIGURE 1-4 MAIN REASONS OF A PROJECTILE DISPERSION

1.4 CATEGORIES OF WEAPONS

In order to see where the projectiles are, whose flight dynamics will be studied in this book, here is a list of the whole range of weapons. They are divided into four categories Figure 1-5.

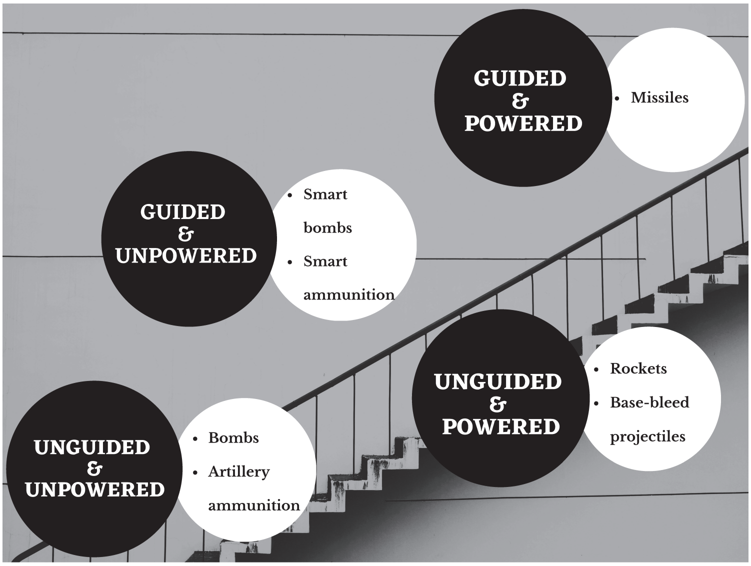


FIGURE 1-5 CATEGORIES OF WEAPONS

The first category includes ballistic weapons such as air flight bombs and artillery shells. The second category (unguided and propelled) contains rocket-powered projectiles fired from the aircraft and ground and gun-fired rocket-assisted projectiles. The third category includes those weapons that are called "smart bombs and ammunition". The fourth category is the one that is referred usually to guided weapons.

1.5 FUTURE DEVELOPMENTS

Exterior ballisticians have traditionally focused their efforts on unguided, rigid, rotationally symmetric, single-body projectiles. Future work in this field will expand into the following areas:

- manoeuvring ("smart") projectiles, with the correction of flight (there are examples of free-flight rockets which are modified with simplified guidance and control section in order to improve accuracy and to extend the range);
- high angle of attack flight;
- sub-munitions dispensing and dispersal;
- aeroballistics of complex multi-body systems.

2 Environment

2.1 Introduction

The environment has a significant impact on the flight of the projectile. Environmental characteristics that influence flight are Earth's shape and gravitation, and the atmosphere. For a better understanding of the influence, we will start from a generalised equation of the motion written for the centre of mass of the projectile:

$$\frac{d(m\boldsymbol{V})}{dt} = \sum \boldsymbol{F} = \boldsymbol{F}_a + \boldsymbol{F}_t + \boldsymbol{G} \tag{2.1}$$

where:

$\boldsymbol{F}_a$ vector of aerodynamic force

$\boldsymbol{F}_t$ thrust vector

$\boldsymbol{G}$ gravity force

From the Equation (2.1), it is clear that Earth's gravitational field directly impacts the flight. On the other side, atmospheric properties, density, temperature, pressure, etc., indirectly influence flight through aerodynamic force and thrust force.

In general, we can present aerodynamic force as the surface integral of pressure, which leads to the conclusion that aerodynamic force is proportional to dynamic pressure $\frac{\rho V^2}{2}$ and further to air density ρ. Furthermore, from fluid dynamics, it is known that the Mach number $M = \frac{V}{a}$, the ratio between the velocity relative to the air and speed of sound, defines the flow regime. The speed of sound depends on temperature T, which is another atmospheric property that indirectly influences flight.

The static pressure p will affect flight through thrust force due to the difference between atmospheric pressure and static pressure at the nozzle exit plane.

If we go back to the equation of motion, we should emphasise that velocity is absolute velocity, including the projectile movement and movement of the atmosphere (wind).

The Equation (2.1) is a differential equation written for the centre of mass of the projectile. Velocity properties can be solved in the first integration of the equation, and to calculate position, we will need second integration. For both integrations, we should consider that both the projectile and Earth have movement. Additionally, to calculate the position of the projectile, we will need to take a point on the Earth as a reference, which will introduce the shape of the Earth as well.

2.2 EARTH

The Earth performs a complex movement consisting of the following movements:

- Rotation around its own axis from west to east, with a period of 23 h 56 min 4.091s, which is equal to 86,164.091s mean solar time. Day length of 24 h represents the relative angular velocity of the Earth relative to the Sun, which encompasses the rotation around its axis and the rotation in orbit around the Sun. The Earth's rotational speed is equal: $\omega_e = \dfrac{2\pi}{86164.091} = 7.292115 \cdot 10^{-5} \dfrac{rad}{s}$
- The vector of the Earth's angular velocity has the direction of the axis of rotation and is directed from the South to the North Pole.
- With an average orbital speed of 100,000 km/h, annual rotation around the Sun has a period of 365 days, 6 hours and 9 minutes.
- Nutational movement of the Earth's axis has a period of about 18.6 years and amplitude that does not exceed 9.2".
- The precessional motion of the Earth's axis is a consequence of the influence of the Moon's gravity, which tends to move the Earth's axis perpendicularly to the plane of its orbit. As a result, the Earth's axis describes a circle of 23.5° with a period of about 26,000 years.
- Movement together with the solar system in relation to other stars.

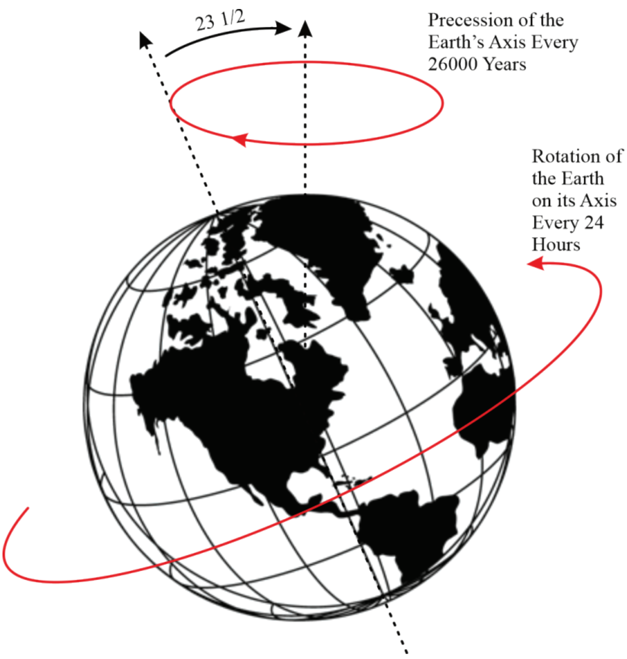


FIGURE 2-1 THE EARTH'S MOVEMENT

When determining the flight of a projectile, all these complex movements of the Earth (except the first one - rotation around its axis) have a negligible influence. Impacts associated with Earth's rotation play a very large role in the dynamics of medium- and long-range ballistic missiles.

2.2.1 EARTH'S SHAPE

The simplest approximation of the Earth's shape is a flat Earth. However, it is known that Earth is not flat, but for short distances, where the curvature of the Earth does not impact the line of sight, flat Earth approximation is usually accurate enough.

The following approximation of the Earth's shape widely used is spherical Earth with a radius of $R_e = 6371 km$. As the Earth's shape is closer to an ellipsoidal model, this spherical Earth approximation accuracy will depend on latitude. Nevertheless, this approximation can give accurate results for ranges less than $1000\ km$. However, a more precise approximation is required for longer ranges.

The surface of the Earth with all its irregularities is called the physical surface of the Earth. Unfortunately, the Earth's physical surface is practically impossible to describe mathematically. Therefore, it is necessary to determine the body closest in shape and size to the Earth and whose surface could be described mathematically.

The geoid (Figure 2-2) represents the closest description of the real Earth's surface of all the geometric bodies. If the Earth were a fluid body, with the same mass distribution, under the action of the Earth's rotation, its free surface would be a geoid. The geoid represents the equipotential surface of the Earth's gravitational field, which coincides with the undisturbed mean sea level. This area extends below the continent. The direction of the local "effective" gravitational field is everywhere along the normal to the surface of the geoid. The mean sea level determines the size of the geoid. The geoid is the surface that is perpendicular to gravity's direction at all points. This is a non-centric gravitational field used to analyse intercontinental flights and satellite trajectories.

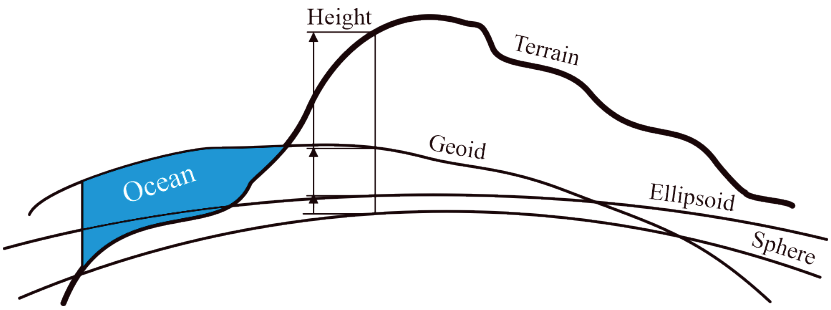


FIGURE 2-2 THE EARTH SHAPE COMPARISON

The reference ellipsoid is the simplest mathematical figure closest to the geoid. This figure was obtained by rotating the ellipse around a minor axis, which corresponds to the Earth's axis of rotation. The result is a spheroid or a rotating ellipsoid. This approximation is defined in detail with a World Geodetic System - WGS 84.

The coordinate origin of WGS 84 (Figure 2-3) is meant to be located at the Earth's centre of mass; the uncertainty is believed to be less than 2 cm.

The WGS 84 meridian of zero longitude is the IERS Reference Meridian, 5.3 arc seconds or 102 metres east of the Greenwich meridian at the latitude of the Royal Observatory.

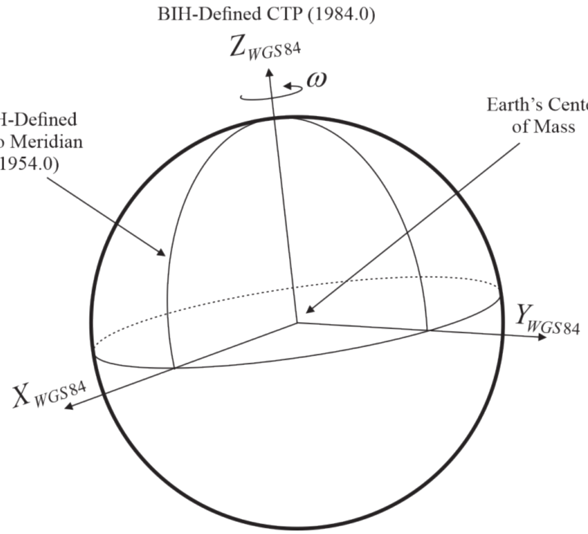


FIGURE 2-3 WGS 84 REFERENCE FRAME

The WGS 84 datum surface is an oblate spheroid with an equatorial radius $a = 6378137\ m$ at the equator and flattening $f = 1/298.257223563$. The refined value of the WGS 84 gravitational constant (mass of Earth's atmosphere included) is $GM = 3986004.418 \cdot 10^8\ \frac{m^3}{s^2}$. The angular velocity of the Earth is defined to be $\omega_e = 72.92115 \cdot 10^{-6}\ \frac{rad}{s}$.

This leads to several computed parameters such as the polar semi-minor axis b, which equals $b = a \cdot (1 - f) = 6356752.3142\ m$, and the first eccentricity squared $e^2 = 6.69437999014 \cdot 10^{-3}$.

wgs84Ellipsoid	*Reference ellipsoid for World Geodetic System 1984*
earthRadius	*Mean radius of planet Earth*
rcurve(ellipsoid,lat)	*Ellipsoidal radii of curvature, returns the parallel radius of curvature at the latitude lat for a reference ellipsoid defined by ellipsoid,*
geocentricLatitude(phi,F)	*Convert geodetic to geocentric latitude, returns the geocentric latitude corresponding to geodetic latitude phi on an ellipsoid with flattening F*
geodeticLatitudeFromGeocentric (phi,F)	*Convert geocentric to geodetic latitude, returns the geodetic latitude corresponding to geocentric latitude psi on an ellipsoid with flattening F*

TABLE 2-1 MATLAB MAPPING TOOLBOX FUNCTIONS THAT REPRESENT SIZE AND SHAPE OF THE EARTH

2.2.2 The Earth's coordinates

The equator is an imaginary circle on the surface of a sphere whose centre coincides with the sphere's centre. The great circles that pass through the North and South poles are called meridians, or lines of longitude. A meridian can be defined at any point on the Earth's surface. The primary meridian, from which the eastern and western meridians are measured, is the meridian that marks the position of the Royal Observatory in Greenwich, England. Longitude (λ) is expressed in degrees, minutes and seconds per arc from 0 to 180 degrees East or West of the base meridian. The equator is the starting point from which the northern and southern coordinates on the Earth's surface are measured. Circles in parallel planes measured North and South relative to the equator are called parallels, or latitude lines. Latitude (φ) is also expressed in degrees, minutes and seconds per arc relative to the centre of the Earth.

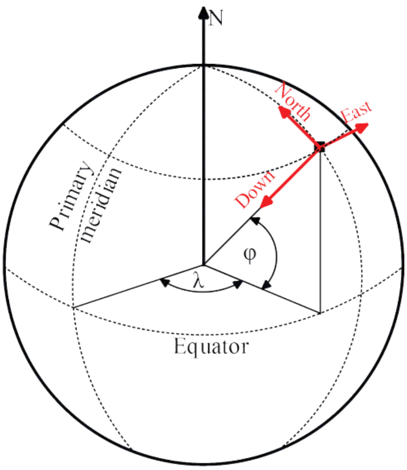


FIGURE 2-4 THE EARTH'S COORDINATES

2.2.3 GEOPOTENTIAL AND GEOMETRIC ALTITUDE

Geopotential $\Phi(x, y, z)$ is a characteristic value for any point having coordinates x, y, z. Thus, the surface defined by the Equation $\Phi(x, y, z) = const$ is called a geopotential surface. The following formula can determine the geopotential:

$$d\Phi = gdh = g_n dH$$
$$\Phi = \int_0^h g(h)\,dh \qquad (2.2)$$
$$H = \int_0^h \frac{g(h)}{g_n} dh = \frac{1}{g_n}\int_0^h g(h) dh = \frac{1}{g_n}\Phi$$

where

g acceleration of free fall

h geometric altitude

Geopotential altitude H is obtained by dividing the geopotential Φ by the standard acceleration of free fall g_n (acceleration at the mean sea level).

$$H = \frac{\Phi}{g_n} = \frac{1}{g_n}\int_0^h g(h)\,dh \qquad (2.3)$$

Where g_n is a standard acceleration of free fall at latitude $\varphi = 45^{\circ}32'33''$

`geoidheight(latitude,longitude)`	*Calculates the geoid height using the EGM96 Geopotential Model*

TABLE 2-2 MATLAB AEROSPACE TOOLBOX FUNCTION

2.2.4 DISTANCE ON THE EARTH

Earth models primarily affect the calculation of range on the trajectory, and generally distance between two points on the Earth.

We can use few approximations of the Earth's shape depending on the range, as explained previously. We will present here a mathematical model and calculation of the distance between two points and the Azimuth angle for Flat Earth, Spherical Earth, and WGS 84.

Flat Earth model

Distance between two points can be calculated based on vector calculation and the Pythagorean theorem.

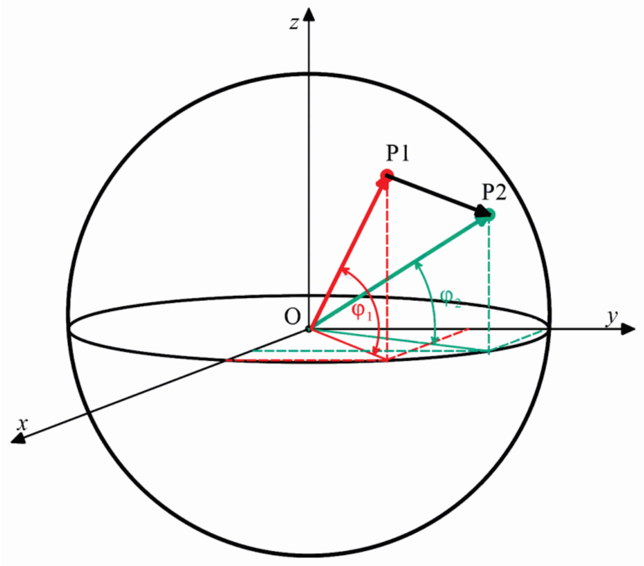


FIGURE 2-5 FLAT EARTH MODEL

Earth coordinates of point P1: $(\varphi_1, \lambda_1, h_1)$

Earth coordinates of point P2: $(\varphi_2, \lambda_2, h_2)$

Vector OP1:

$$\mathbf{OP1} = R_e \begin{bmatrix} \cos\varphi_1 \cos\lambda_1 & \cos\varphi_1 \sin\lambda_1 & \sin\varphi_1 \end{bmatrix} \tag{2.4}$$

Vector OP2:

$$\mathbf{OP2} = R_e \begin{bmatrix} \cos\varphi_2 \cos\lambda_2 & \cos\varphi_2 \sin\lambda_2 & \sin\varphi_2 \end{bmatrix} \tag{2.5}$$

Distance vector:

$$\mathbf{P1P2} = \mathbf{OP2} - \mathbf{OP1} = \begin{bmatrix} x_d & y_d & z_d \end{bmatrix} \tag{2.6}$$

Distance:

$$D_{P1P2} = \sqrt{\left(x_d^{\ 2} + y_d^{\ 2} + z_d^{\ 2}\right)} \tag{2.7}$$

Spherical Earth model – the Haversine formula

The Haversine formula determines the great-circle distance between two points on a sphere, given their longitudes and latitudes.

Earth coordinates of point P1: $(\varphi_1, \lambda_1, h_1)$

Earth coordinates of point P2: $(\varphi_2, \lambda_2, h_2)$

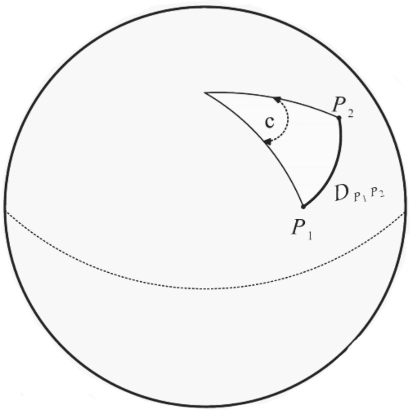


FIGURE 2-6 SPHERICAL EARTH MODEL – THE HAVERSINE FORMULA

Let the central angle, c between any two points on the sphere be:

$$c = \frac{D}{R_e} \tag{2.8}$$

where D is the distance between points.

The haversine formula allows the haversine of c (that is, $hav(c)$) to be computed directly from the latitude (represented by φ) and longitude (represented by λ) of the two points:

$$hav(c) = hav(\varphi_2 - \varphi_1) + \cos(\varphi_1)\cos(\varphi_2)hav(\lambda_2 - \lambda_1) \tag{2.9}$$

where:

$$hav(c) = \sin^2\left(\frac{c}{2}\right) = \frac{1 - \cos(c)}{2} \tag{2.10}$$

We can solve distance by combining Equations (2.8) to (2.10):

$$\begin{aligned} D_{P1P2} &= 2R_e \arcsin\left(\sqrt{hav(c)}\right) = \\ &= 2R_e \arcsin\left(\sqrt{\sin^2\left(\frac{\varphi_2-\varphi_1}{2}\right)+\cos(\varphi_2)\cos(\varphi_1)\sin^2\left(\frac{\lambda_2-\lambda_1}{2}\right)}\right) \end{aligned} \tag{2.11}$$

Spherical Earth model –equirectangular approximation

The Pythagoras' theorem can be used for small distances on an equirectangular projection instead of the Haversine formula.

North projection:

$$N = R_e(\varphi_1 - \varphi_2)\frac{\pi}{180} \tag{2.12}$$

East projection:

$$E = R_e(\lambda_2 - \lambda_1)\frac{\pi}{180}\cos\left(\frac{\varphi_1+\varphi_2}{2}\right) \tag{2.13}$$

Distance:

$$D_{P1P2} = \sqrt{N^2 + E^2} \tag{2.14}$$

Bearing:

$$\psi = \arctan\left(\frac{E}{N}\right) \tag{2.15}$$

WGS84 model – distance on the ellipsoid

An ellipsoid approximates Earth's surface much better than a sphere, or a flat surface does. The shortest distance along the surface of an ellipsoid between two points on the surface is along the geodesic. Finding the geodesic between two points on the Earth, the so-called inverse geodetic problem, was the focus of many mathematicians and geodesists throughout the 18th and 19th centuries, with major contributions by Clairaut, Legendre, Bessel, and Helmert.

Methods for computing the geodesic distance are widely available in geographical information systems, software libraries, standalone utilities, and online tools. The most commonly used algorithm is by Vincenty, who uses a series that is accurate to the third order in the flattening of the ellipsoid, i.e., about 0.5 mm; however, the algorithm fails to converge for nearly antipodal points. This defect is cured in the algorithm given by Karney, who employs series which are accurate to the sixth order in the flattening. This results in a precise algorithm to full double precision and converges for arbitrary pairs of points on the Earth.

The first (direct) method computes the location of a point that is a given distance and azimuth (direction) from another point. The second (inverse) method computes the geographical distance and azimuth between two given points. They have been widely used in geodesy because they are accurate to within 0.5 mm on the Earth ellipsoid.

Matlab Mapping toolbox is one of the tools that can be used for distance calculation. Related functions are presented in Table 2-4.

`[arclen,az]=distance(lat1,lon1, lat2,lon2,ellipsoid)`	*Distance and azimuth between points on sphere or ellipsoid*
`az=azimuth(lat1,lon1,lat2,lon2, elipsoid)`	*Azimuth between points on sphere or ellipsoid*
`[latout,lonout]=reckon(lat,lon, arclen,az,ellipsoid)`	*Point at specified azimuth, range on sphere or ellipsoid*

TABLE 2-3 MATLAB MAPPING TOOLBOX FUNCTIONS FOR DISTANCE CALCULATIONS

2.2.5 GRAVITY

Gravity is the vector sum of the gravitational attraction and the centrifugal force induced by the Earth's rotation. It is, therefore, a complicated function of latitude and the radial distance from the centre of the Earth. However, allowance can be made for centrifugal forces. The acceleration of the free fall $g_\varphi(h)$ may be found for each height and latitude by use of the nominal value of Earth's radius r_φ at each latitude with Newton's law of gravitation:

$$g_\varphi(h) = g_{0\varphi}\left(\frac{r_\varphi}{r_\varphi + h}\right)^2 \tag{2.16}$$

where

r_φ the normal radius of the Earth at a specific latitude

$g_{0\varphi}$ the acceleration of free fall at sea level for latitude φ.

Gravity (acceleration of free fall) at sea level $g_{0\varphi}$ for latitudes other than 45^o can be obtained by Lambert's Equation, in which gravity varies with latitude φ:

$$g_{0\varphi} = 9.80616\left(1 - 0.0026373\cos 2\varphi - 0.0000059\cos^2 2\varphi\right)\quad\left[\frac{m}{s^2}\right] \tag{2.17}$$

Values from this relationship and nominal Earth's radius for different latitude values are given in Table 2-4.

After substituting equation (2.16) into (2.3) and integration, the relationship between the geopotential and geometric altitude is given by the following expression:

$$H = \frac{r_\varphi h}{r_\varphi + h} \cdot \frac{g_{0\varphi}}{g_n} \tag{2.18}$$

Latitude	Acceleration of free fall	Nominal Earth's radius
15° N	9.78381	6337.84
30° N	9.79324	6345.65
45° N	9.80665	6356.77
60° N	9.81911	6367.10
80° N	9.83051	6376.56

TABLE 2-4 GRAVITY ACCELERATION WITH RESPECT TO LATITUDE

`gravitywgs84(h,lat)`	*Implement 1984 World Geodetic System (WGS84) representation of Earth's gravity*

TABLE 2-5 MATLAB AEROSPACE TOOLBOX GRAVITY FUNCTION

2.3 Atmosphere

2.3.1 Basis of standard atmosphere

The international standard presents information on the seasonal, latitudinal, longitudinal, and day-to-day variability of atmospheric properties at levels between the surface and 80 km.

Parameters of the atmosphere for the nominal trajectory calculation are based on the standard atmosphere.

Some special considerations employed in the development of this family of reference atmospheres are listed below.

- Except for the 15° latitude model, the reference atmospheres are considered applicable to the northern hemisphere only. However, it is believed that they closely approximate mid-latitude conditions in the southern hemisphere.
- The models are defined by temperature-altitude profiles in which the vertical gradients of temperature are constant with respect to geopotential altitude within each of a number of layers.
- The air is assumed to be a perfect gas, free from moisture and dust.
- The tables of the ISO Standards Atmosphere have been calculated assuming the air to be a perfect gas free from moisture and dust and based on conventional initial values of temperature, pressure, and density of the air for mean sea level. The following constants and characteristics are used for calculations, and their numerical values are given in Table 2-6

Nomenclature:

g_n	standard acceleration of free fall. It conforms with latitude $\varphi = 45^{\circ}32'33''$ using Lambert's Equation of the acceleration of free fall as a function of latitude φ Equation (2.17)
p_n	standard air pressure
R^*	universal gas constant
R	specific gas constant
T_0	thermodynamic ice-point temperature at mean sea level
T_n	standard thermodynamic air temperature at mean sea level
t_0	Celsius ice-point temperature at mean sea level
t_n	standard Celsius air temperature at mean sea level
ρ_n	standard air density

Symbol	Value	Unit of measurement
g_n	9.80665	ms^{-2}
p_n	101,325 1,013.250 760	Pa $mbar$ $mmHg$
R^*	8,314.32	$J \cdot K^{-1} \cdot kmol^{-1}$ or $kg \cdot m^2 \cdot s^{-2} \cdot K^{-1} \cdot kmol^{-1}$
R	287.05287	$J \cdot K^{-1} \cdot kg^{-1}$ or $m^2 \cdot K^{-1} \cdot s^{-2}$
T_0	273.15	K
T_n	288.15	K
t_0	0.00	^oC
t_n	15.00	^oC
ρ_n	1.225	$kg \cdot m^{-3}$

TABLE 2-6 MAIN CONSTANT AND CHARACTERISTIC ADOPTED FOR THE CALCULATION OF THE ISO STANDARD ATMOSPHERE

2.3.2 TEMPERATURE AND VERTICAL TEMPERATURE GRADIENT

Thermodynamic temperature for the melting point of ice under the pressure of 101.325,0 Pa is taken as $T_0 = 273{,}15\,\mathrm{K}$. Thermodynamic temperature T (in kelvins, K) is:

$$T = T_0 + t \tag{2.19}$$

Where t is the Celsius temperature.

According to the temperature variations with altitude, the atmosphere is divided into several layers: the troposphere, the stratosphere, and the mesosphere, Figure 2-7.

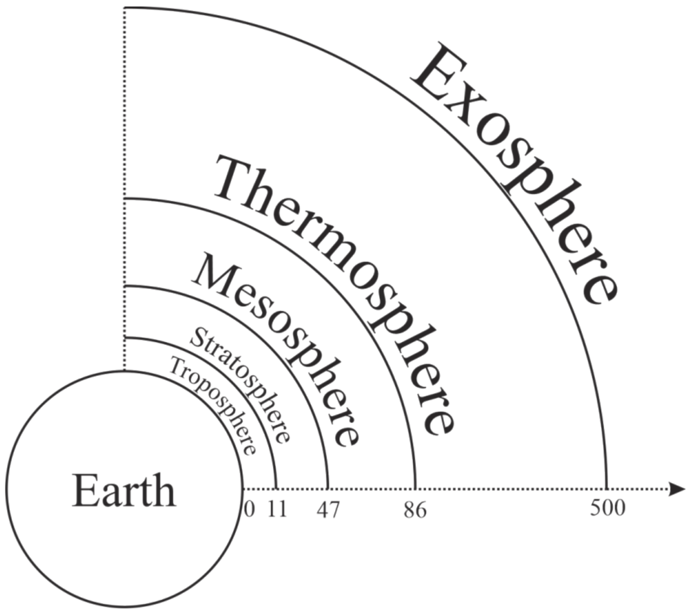


FIGURE 2-7 ATMOSPHERE LAYERS

The transitional zones between these layers are called the tropopause, stratopause, and mesopause.

For calculating a standard atmosphere, the temperature of each layer is taken as a linear function of geopotential altitude so that:

$$T = T_b + \beta\left(H - H_b\right) \tag{2.20}$$

Where T_b and H_b are respectively the temperature and the geopotential altitude of the lower limit of the layer concerned and β is the vertical temperature gradient $\frac{dT}{dH}$.

The values of temperature and its vertical gradients adopted for the ISO Standard Atmosphere are given in Table 2-7. Diagram of change of temperature with height is given in Figure 2-8

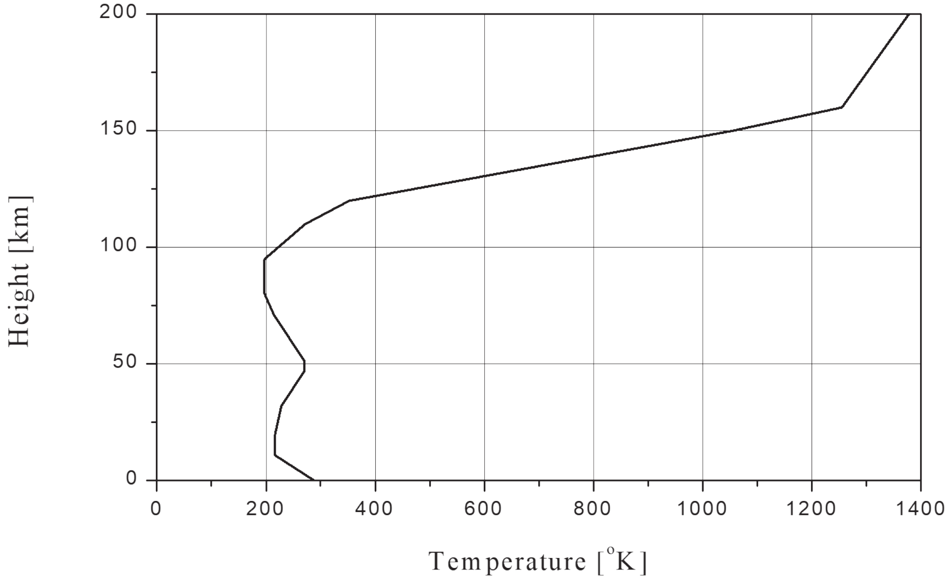


FIGURE 2-8 TEMPERATURE AND ITS GRADIENT VS ALTITUDE

Geopotential altitude at the beginning of layers $H_b\,[km]$	**Temperature gradient** $\beta = \frac{dT}{dH}[K/km]$	**Atmospheric temperature lower limit of the layer** $T_b\,[K]$	**Atmospheric pressure at the beginning of layers** $p_b\,[Pa]$
0.0	-6.5	$T_{b0} = 288.15$	$p_{b0} = 101325$
11.0	0.0	$T_{b11} = 216.65$	$p_{b11} = p_{b0}\left[1-71.5/T_{b0}\right]^{5.25588}$
20.0	1.0	$T_{b20} = \mathrm{T}_{b11} = 216.65$	$p_{b20} = p_{b11}.e^{-\frac{307.46897}{T_{b11}}}$
32.0	2.8	$T_{b32} = 228.65$	$p_{b32} = p_{b20}\left(1+\frac{12}{T_{b20}}\right)^{-34.16322}$

TABLE 2-7 VARIATION OF TEMPERATURE, TEMPERATURE GRADIENT AND PRESSURE AT THE BEGINNING OF LAYERS VS. GEOPOTENTIAL ALTITUDE

`[T,a,P,Rho]=atmoscoesa(h)`	*Implements the mathematical representation of the 1976 COESA United States standard lower atmospheric values. These values are absolute temperature, pressure, density, and speed of sound for the input geopotential altitude. Below the geopotential altitude of 0 m and above the geopotential altitude of 84,852 m, the function extrapolates values. It extrapolates temperature values linearly and pressure values logarithmically.*
`[T,a,P,Rho]=atmosisa(h)`	*Implements the mathematical representation of the International Standard Atmosphere values for ambient temperature, pressure, density, and speed of sound for the input geopotential altitude. This function assumes that below the geopotential altitude of 0 km and above the geopotential altitude of the tropopause, temperature and pressure values are held.*

TABLE 2-8 MATLAB AEROSPACE TOOLBOX ATMOSPHERE MODELS

2.3.3 THE HYDROSTATIC EQUATION AND THE PERFECT GAS LAW

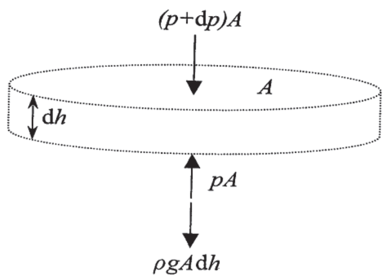


FIGURE 2-9 A SLICE OF ATMOSPHERE IN HYDROSTATIC EQUILIBRIUM

Being static with respect to the Earth, the atmosphere is subject to gravitational forces. Therefore, the conditions of the air in static equilibrium are specified by the hydrostatic equation, which relates air pressure p, density ρ, acceleration of free fall g, and geometric altitude h as follows:

$$-dp = \rho g \, dh \tag{2.21}$$

The perfect gas law relates air pressure to density and temperature as follows:

$$p = \frac{\rho R^* T}{M} \tag{2.22}$$

At the altitudes of interest, $\frac{R^*}{M} = const. = R$; hence:

$$p = \rho RT \tag{2.23}$$

The specific gas constant of dry air R is equal: $R = 287.05287\left[\frac{J}{kg\,K}\right]$. The air is assumed to be perfect gas, free from moisture and dust ($M = 28.964420\ \frac{\text{kg}}{\text{kmol}}$).

2.3.4 PRESSURE

Assuming a linear variation of the temperature with geopotential altitude, the simultaneous solution of the Equation of static atmosphere (2.21) and the perfect gas law (2.22) yields the following expression for pressure:

$$\ln p = \ln p_b - \frac{g_n}{\beta R}\ln\frac{T_b + \beta(H - H_b)}{T_b} \tag{2.24}$$

or

$$p = p_b\left[1 + \frac{\beta}{T_b}(H - H_b)\right]^{-g_n/\beta R} \tag{2.25}$$

and

$$\ln p = \ln p_b - \frac{g_n}{RT}(H - H_b) \tag{2.26}$$

or

$$p = p_b \exp\left[-\frac{g_n}{RT}(H - H_b)\right] \text{ for } \beta = 0 \tag{2.27}$$

Here subscript "b" refers to the values of the pertinent characteristics to the lower limit of the layer concerned.

2.3.5 DENSITY AND SPECIFIC WEIGHT

The density ρ is calculated from the pressure and the temperature using the perfect gas law:

$$\rho = \frac{p}{RT} = 0.003483679\frac{p}{T}\ \left[kg/m^3\right] \tag{2.28}$$

The specific weight γ is the weight per unit volume of air, that is:

$$\gamma = \rho g \tag{2.29}$$

Speed of sound

$$a = \sqrt{kRT} = 20.046796\sqrt{T} \ \left[m/s\right]$$
$$k = \frac{C_p}{C_v} = 1.41 \tag{2.30}$$

2.3.6 Effect of the air humidity

The humidity has a negligible effect on the air density and the speed of sound. However, in order to obtain accurate correction of the fire, it is useful to take into the expression air density and speed of sound. The humidity of air can be represented in three ways:

- mass ratio, r ;
- the partial pressure of water vapour, e' ;
- dew temperature, t_d .

The quantity q is the ratio of water vapour mass m_v , to the mass of dry air m_a , in the same volume:

$$r = \frac{m_v}{m_a} \tag{2.31}$$

Where m_v is the mass of water vapour in grams, and m_a is the mass of dry air in kilograms.

The total pressure of (air + water vapour) is:

$$p = p_a + e' \tag{2.32}$$

The state equations for dry air and water vapour are:

$$p_a V = m_a RT \tag{2.33}$$

$$e'V = m_v 10^{-3} R_v T \tag{2.34}$$

By using Equations (2.31)-(2.34) we obtain partial pressure of water vapour, which is:

$$e' = \frac{r}{\frac{R \cdot 10^3}{R_v} + r} p \tag{2.35}$$

Where $\frac{R \cdot 10^3}{R_v}$=621.98, e' and p are expressed in mbar.

The pressure of water vapour in saturated air corresponds to the mass ratio r_w and is given by:

$$e'_w = \frac{r_w}{\frac{R \cdot 10^3}{R_v} + r_w} p \tag{2.36}$$

The partial pressure e'_w depends on the air temperature. For the interval $-20^0\,\mathrm{C} < t < 30^0\,\mathrm{C}$, we have e'_w in mbar

$$\begin{aligned} &e'_w = 6.107 \cdot 10^{\frac{at}{b+t}} \\ &a = 7.5 \quad b = 237.3 \quad \text{for} \quad t > 0^0\,\mathrm{C} \\ &a = 9.5 \quad b = 265.5 \quad \text{for} \quad t < 0^0\,\mathrm{C} \end{aligned} \tag{2.37}$$

The dew air temperature is approximated well:

$$t_d = \frac{237.3 \log_{10} \frac{e'_w}{6.1070}}{7.5 - \log_{10} \frac{e'_w}{6.1070}} \tag{2.38}$$

The relative humidity in percents is:

$$R_H = \left(\frac{e'}{e'_w}\right)_{(p,t)} \cdot 100 \tag{2.39}$$

Since $R_v \approx \frac{8}{5} R$, Equation (2.34) may be written as:

$$\frac{5}{8} e' V = m_v \cdot 10^{-3} RT \tag{2.40}$$

Having in mind that:

$$p_a = p - e' \tag{2.41}$$

$$\rho = \frac{m_a + m_v \cdot 10^{-3}}{V} \tag{2.42}$$

From Equations (2.33) to (2.41) we develop:

$$\rho = \frac{p}{R \frac{T}{1 - \frac{3}{8} \frac{e'}{p}}} \tag{2.43}$$

Introducing the “fictitious” temperature:

$$\tau = \frac{T}{1 - \frac{3}{8}\frac{e'}{p}} \tag{2.44}$$

Equation (2.43) becomes:

$$\rho = \frac{p}{R\tau} \tag{2.45}$$

The real air (with water vapour) can be assumed as the ideal gas in the calculation of the air density, if the “fictitious” temperature is substituted instead of the absolute temperature.

The speed of sound can be calculated by using the previous formula with the “fictitious” temperature:

$$a = \sqrt{kR\tau} = 20.046796\sqrt{\tau} \ \left[\mathrm{m/s}\right] \tag{2.46}$$

For the given parameters of the air – pressure (*p*), temperature ($t^0\mathrm{C}$ or T in K) and relative humidity (R_H), the air density and speed of sound are determined as follows:

- the partial pressure of water vapour (e'_w) of saturated air is calculated by using Equation (2.37);
- the real partial pressure of water vapour (e') is determined by using Equation (2.40);
- After calculating the “fictitious” temperature by Equation (2.44), the air density and speed of sound are determined by Equations (2.45) and (2.46), respectively.

2.3.7 WIND

The wind is the most prevalent and complex natural disturbance to projectile flight. Changing wind fields have a number of meteorological origins. Wind models produce force and moments input through the projectile’s aerodynamics.

The wind is a vector field that varies in space and time; its magnitude may approach the rocket projectile’s velocity while leaving the launcher. The wind generally may be modelled as:

$$\mathbf{w}(\mathbf{r},t) = \begin{bmatrix} W_x(\mathbf{r},t) \\ W_y(\mathbf{r},t) \\ W_z(\mathbf{r},t) \end{bmatrix} \tag{2.47}$$

Where **r** is the vector of projectile’s position and *t* is the time of flight.

Atmospheric turbulence is a random variable that is superimposed on the slowly varying mean wind field:

$$\mathbf{w}(\mathbf{r},t) = \mathbf{w}_0(\mathbf{r},t) + \Delta\mathbf{w}(\mathbf{r},t) \tag{2.48}$$

$\mathbf{w}_0(\mathbf{r},t)$ characterised a component that measurably persists over some time and region. While the statistical properties of $\Delta\mathbf{w}(\mathbf{r},t)$ may be relatively constant, the actual wind-speed variations occur probabilistically and, therefore, are not strictly predictable. $\Delta\mathbf{w}(\mathbf{r},t)$ is a random process whose components are characterised by probability distributions, expected values, and power spectral density, which are discussed in the chapter on projectile dispersion.

The turbulence may be described as a wavelike phenomenon whose component amplitudes vary with frequency. The von Karman turbulence spectra are widely accepted representations of atmospheric turbulence in which the incremental wind "power" $\Phi(\omega)$ ($\mathrm{m}^2/\mathrm{s}^2$) is a function of wavelength or wave frequency ω (rad/m). For example, a one-dimensional spectrum for disturbances that are transverse to the flight direction is:

$$\Phi(\omega) = \frac{\sigma^2 L}{2\pi} \frac{\left[1 + \frac{8}{3}(aL\omega)^2\right]}{\left[1 + (aL\omega)^2\right]^{\frac{11}{6}}} \tag{2.49}$$

Where $\sigma, a \approx 1.4, L \approx 1.750\mathrm{m}$ are constants of the random process. This spectrum is not easily modelled in state-space format. Though a less accurate fit to experimental data, the Dryden model does not have such a problem for numerical simulation. For transverse turbulence, the Dryden spectrum takes the form:

$$\Phi(\omega) = \frac{\sigma^2 L}{\pi} \frac{\left[1 + 3(L\omega)^2\right]}{\left[1 + (L\omega)^2\right]^2} \tag{2.50}$$

The spectrum for axial turbulence (i.e. along the line of flight) is:

$$\Phi(\omega) = \frac{2\sigma^2 L}{\pi} \frac{1}{\left[1 + (L\omega)^2\right]^2} \tag{2.51}$$

When significant features of the wind field are large compared to the vehicle size (very often the case with most projectiles and missiles), Equation (2.47) is considered as exogenous or external disturbances to the vehicle's dynamics, neglecting the converse effect of the vehicle's presence on the wind field itself. However, this may not be true for aircraft flight.

3 Reference Frames & Transformations

3.1 INTRODUCTION

In physics studies, the establishment of two systems is fundamental: one is the reference frame of a system relative to the observer, and another is the coordinate system.

A reference frame (or simply "frame") is specified by an ordered set of three mutually orthogonal, possibly time-dependent, unit-length direction vectors. A reference frame has an associated centre.

A coordinate system specifies a mechanism for locating points within a reference frame.

When producing or using state (position and velocity) or orientation (pointing) data, one needs to understand both the reference frame and the coordinate system being used.

Fundamental to flight mechanics and inertial navigation is the precise definition of several reference frames. Each frame is an orthogonal, right-handed, coordinate frame or set axis.

Types of reference frames:

- An inertial frame of reference is a frame of reference that is not undergoing acceleration or rotation. In an inertial frame of reference, a physical object with zero net force acting on it moves with a constant velocity (which might be zero) - or, equivalently, it is a frame of reference in which Newton's first law of motion holds.
- A non-inertial reference frame is a frame of reference that undergoes acceleration with respect to an inertial frame. An accelerometer at rest in a non-inertial frame will, in general, detect non-zero acceleration. While the equations of motion are the same in all inertial frames, in non-inertial frames, they vary from frame to frame depending on the acceleration and rotation.

For navigation over the Earth, it is necessary to define axis sets that allow the inertial measurements to be related to the basic directions of the Earth.

3.2 NOTATION AND TRANSFORMATION

Let F_v and F_b be two right-handed reference frames; in a particular case, it may be missile carried vertical and body frames. In the general case, two frames have relative motion, both linear and angular.

Consider now the description of a typical vector that does not depend on the motion of the frame. For example, the vector $\boldsymbol{v}$ may be presented by projections in F_v or F_b frame:

$$\mathbf{v}^v = \begin{bmatrix} v_{x_v} \\ v_{y_v} \\ v_{z_v} \end{bmatrix}, \ \mathbf{v}^b = \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} \tag{3.1}$$

where v_{x_v}, v_{y_v} and v_{z_v} are projections of vector $\boldsymbol{v}$ on the axes of F_v frame, and v_x, v_y and v_z are projections of the same vector on the axes of F_b frame. The upper index denotes in which frame the projections of the vector are given.

The angular velocity of F_b frame with respect to F_v frame may be presented by:

- vector $\boldsymbol{\omega}_{vb}$
- or matrix column $\boldsymbol{\omega}_{vb}^{v}$ with components in F_v frame
- or matrix column $\boldsymbol{\omega}_{vb}^{b}$ with components in F_b frame.

Transformation of a vector defined by matrix column in F_b frame into matrix column with elements in F_v frame is given by matrix multiplication:

$$\mathbf{v}^v = \mathbf{C}_b^v \mathbf{v}^b \tag{3.2}$$

where $\mathbf{C}_b^v$ is the direction cosine matrix, which is a 3x3 matrix, the columns of which represent unit vectors in body axes projected along the axes of F_v reference frame. $\mathbf{C}_b^v$ is written in component form as follows

$$\mathbf{C}_b^v = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix} \tag{3.3}$$

The element of *i*- row and the *j*-column represents the cosine of the angle between the *i*-axis of the F_v frame and the *j*-axis of the body frame.

A rotation matrix is a special orthogonal matrix. By definition, a special orthogonal matrix has these properties:

$$\mathbf{C}_b^v \left(\mathbf{C}_b^v\right)^T = \mathbf{I} \tag{3.4}$$

where $\left(\mathbf{C}_b^v\right)^T$ is the transpose of $\mathbf{C}_b^v$ and I is the identity matrix, and $\det\left(\mathbf{C}_b^v\right) = 1$.

From those properties, we have few other helpful consequences:

1. $\mathbf{C}_b^v$ is normalized: the squares of the elements in any row or column sum to 1.
2. $\mathbf{C}_b^v$ is orthogonal: the dot product of any pair of rows or any pair of columns is 0.
3. The rows of $\mathbf{C}_b^v$ represent the coordinates in the original space of unit vectors along the coordinate axes of the rotated space.
4. The columns of $\mathbf{C}_b^v$ represent the coordinates in the rotated space of unit vectors along the axes of the original space.

A vector defined in body axes, $\mathbf{v}^b$, may be expressed in F_v frame by pre-multiplying the matrix-column by the direction cosine matrix (3.2).

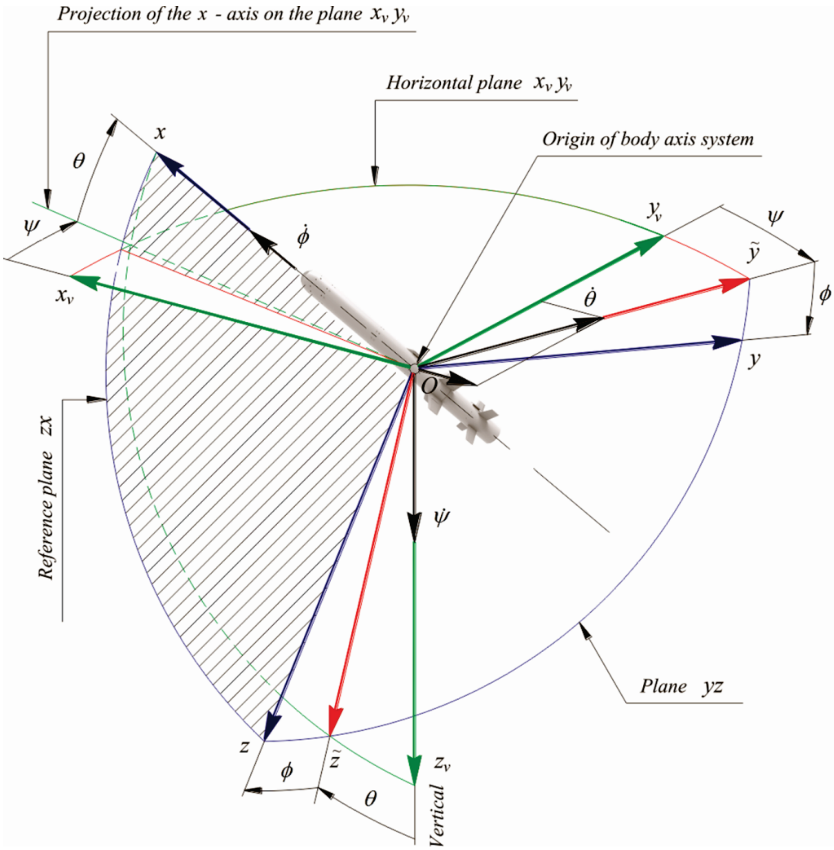


FIGURE 3-1 DEFINITION OF THE BODY AND AEROBALLISTIC REFERENCE FRAMES

The transformation matrix can be given in terms of rotation angles about the three coordinate axes. A transformation from one coordinate frame to another can be carried out as three successive rotations about different axes. Let consider rotations from the missile carried vertical frame (F_v) into the body frame (F_b) shown in Figure 3-1. The frame F_v represents the initial frame and F_b is the frame after three rotations. The three successive rotations are:

- rotation through an angle ψ about z_v - axis
- rotation through an angle θ about new $\tilde{y}$ - axis
- rotation through an angle ϕ about x - axis

These three rotations may be expressed mathematically as three separate direction cosine matrices:

rotation about z_v - axis for ψ

$$\mathbf{C}_1 = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{3.5}$$

rotation about $\tilde{y}$ - axis for θ

$$\mathbf{C}_2 = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \tag{3.6}$$

rotation about x - axis for ϕ

$$\mathbf{C}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \tag{3.7}$$

Thus a transformation from F_v to F_b frame may be expressed as the product of these three separate transformations as follows:

$$\mathbf{C}_v^b = \mathbf{C}_3(\phi)\mathbf{C}_2(\theta)\mathbf{C}_1(\psi) = \mathbf{C}(\psi,\theta,\phi) \tag{3.8}$$

The inverse transformation from the body to F_v frame is given by:

$$\mathbf{C}_b^v = \left(\mathbf{C}_v^b\right)^T = \mathbf{C}_1^T(\psi)\mathbf{C}_2^T(\theta)\mathbf{C}_3^T(\phi) = \mathbf{C}^T(\psi,\theta,\phi) \tag{3.9}$$

Substituting of matrices $\mathbf{C}_1, \mathbf{C}_2, \mathbf{C}_3$ into Equation (3.9) gives:

$$\mathbf{C}_b^v = \begin{bmatrix} c\theta c\psi & -c\phi s\psi + s\phi s\theta c\psi & s\phi s\psi + c\phi s\theta c\psi \\ c\theta s\psi & c\phi c\psi + s\phi s\theta s\psi & -s\phi c\psi + c\phi s\theta s\psi \\ -s\theta & s\phi c\theta & c\phi c\theta \end{bmatrix} \tag{3.10}$$

where $s = \sin$, $c = \cos$.

The general direction cosine matrix (3.8), which transforms a vector from the initial position frame to the final position frame, will be:

$$\mathbf{C}(\psi,\theta,\phi) = \left[\mathbf{C}_b^v(\psi,\theta,\phi)\right]^T \tag{3.11}$$

For small angle rotations

$$\sin\phi \to \phi, \quad \sin\theta \to \theta, \quad \sin\psi \to \psi$$

And the cosines of these angles approach unity. Using these approximations, the direction cosine matrix expressed in terms of the Euler angles is reduced to the skew-symmetric form:

$$\mathbf{C}_b^v = \begin{bmatrix} 1 & -\psi & \theta \\ \psi & 1 & -\phi \\ -\theta & \phi & 1 \end{bmatrix} \tag{3.12}$$

The rate of change of $\mathbf{C}_b^v$ with time is given by:

$$\dot{\mathbf{C}}_b^v = \lim_{\Delta t \to 0} \frac{\Delta \mathbf{C}_b^v}{\Delta t} = \lim_{\Delta t \to 0} \frac{\mathbf{C}_b^v(t+\Delta t) - \mathbf{C}_b^v(t)}{\Delta t} \tag{3.13}$$

Where $\mathbf{C}_b^v(t+\Delta t)$ and $\mathbf{C}_b^v(t)$ represent the direction cosine matrix at times $t+\Delta t$ and t respectively.

The matrix $\mathbf{C}_b^v(t+\Delta t)$ can be written as the product of two matrices:

- the matrix $\mathbf{C}_b^v$ at the time t and
- the matrix $\mathbf{A}(t)$ which relates the b -frame at a time t to the b -frame at a time $t+\Delta t$.

$$\mathbf{C}_b^v(t+\Delta t) = \mathbf{C}_b^v(t)\mathbf{A}(t) \tag{3.14}$$

For small angle rotations, by using approximation (3.12), the matrix $\mathbf{A}(t)$ may be written as follows:

$$\mathbf{A}(t) = [\mathbf{I} + (\Delta\boldsymbol{\psi}\times)] \tag{3.15}$$

Where:

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad (\Delta\boldsymbol{\psi}\times) = \begin{bmatrix} 0 & -\Delta\psi & \Delta\theta \\ \Delta\psi & 0 & -\Delta\phi \\ -\Delta\theta & \Delta\phi & 0 \end{bmatrix} \tag{3.16}$$

The skew-symmetric form $(\Delta\boldsymbol{\psi}\times)$ is written in terms of components of a vector

$$\Delta\boldsymbol{\psi} = [\Delta\phi \quad \Delta\theta \quad \Delta\psi]^T \tag{3.17}$$

where $\Delta\psi, \Delta\theta$ and $\Delta\phi$ are small rotation angles through which the b -frame has rotated over the time interval Δt about its yaw, pitch, and roll axis, respectively.

Substituting for Equation (3.14) in Equation (3.13) we obtain

$$\dot{\mathbf{C}}_b^v = \mathbf{C}_b^v \lim_{\Delta t \to 0} \frac{(\Delta \boldsymbol{\psi} \times)}{\Delta t} \tag{3.18}$$

since

$$\lim_{\Delta t \to 0} \frac{\Delta \boldsymbol{\psi}}{\Delta t} = \begin{bmatrix} \dot{\phi} & \dot{\theta} & \dot{\psi} \end{bmatrix}^T = \boldsymbol{\omega}_{vb}^b \tag{3.19}$$

where

$$\boldsymbol{\omega}_{vb}^b = \begin{bmatrix} \omega_x & \omega_y & \omega_z \end{bmatrix}^T \tag{3.20}$$

represents the turn rate of F_b - frame with respect to the F_v - frame expressed in body axes.

Having in mind the skew matrix written by components of turn rate

$$\left(\boldsymbol{\omega}_{vb}^b \times\right) = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix} \tag{3.21}$$

the rate of change of the direction cosines matrix $\dot{\mathbf{C}}_b^v$ may be written:

$$\dot{\mathbf{C}}_b^v = \mathbf{C}_b^v \left(\boldsymbol{\omega}_{vb}^b \times\right) \tag{3.22}$$

The relative angular velocity $\boldsymbol{\omega}_{vb}^b$ may be expressed in terms of the angular velocities of F_b frame and F_v frame with respect to the inertial frame:

$$\boldsymbol{\omega}_{vb}^b = \boldsymbol{\omega}_{ib}^b - \mathbf{C}_v^b \boldsymbol{\omega}_{iv}^v \tag{3.23}$$

There is another way of writing Equation (3.22) in terms of absolute angular velocities:

$$\dot{\mathbf{C}}_b^v = \mathbf{C}_b^v \left(\boldsymbol{\omega}_{ib}^b \times\right) - \left(\boldsymbol{\omega}_{iv}^v \times\right) \mathbf{C}_b^v \tag{3.24}$$

3.3 EULER ANGLES EXPRESSED IN TERMS OF DIRECTIONS COSINES

The Euler angles may be derived directly from the direction cosines matrix $\mathbf{C}_b^v$ given by (3.10), which transforms a vector defined in F_b frame to a vector defined in F_v frame.

For the conditions where θ is not equal to 90° the Euler angles can be determined by using

$$\begin{aligned} \phi &= \arctan\left(\frac{c_{32}}{c_{33}}\right) \\ \theta &= \arcsin\left(-c_{31}\right) \\ \psi &= \arctan\left(\frac{c_{21}}{c_{11}}\right) \end{aligned} \tag{3.25}$$

If θ approaches to 90°, the Equations in ϕ and ψ become indeterminate because the numerator and denominator approach zero.

For θ near 90° we use

$$\psi - \phi = \arctan\frac{c_{23} - c_{12}}{c_{13} + c_{22}} \tag{3.26}$$

If $\theta \approx 90^\circ$, two axes (x and z) become parallel, i.e., vertical and one degree of freedom is lost. Either ϕ or ψ may be selected arbitrarily to satisfy some other condition while the unspecified angle is chosen to fulfil Equation (3.26). During vertical launch and flight, we usually freeze one angle, ψ for instance, at its current value (zero value), and calculate ϕ by using Equation (3.26).

`angle2dcm(Ang1,Ang2,Ang3, rotationSequence)`	*Create direction cosine matrix from rotation angles*
`dcm2angle(n)`	*Create rotation angles from direction cosine matrix*

TABLE 3-1 AEROSPACE TOOLBOX FUNCTIONS CONVERTING EULER ANGLES AND DCM

3.4 FRAMES AND TRANSFORMATIONS IN FLIGHT MECHANICS

The following coordinate frames are used in flight mechanics:

The Earth inertial frame (**ECI**, Earth-Centered-Inetrtial, i-frame, F_i) has its origin at the centre of the Earth and axes, which are nonrotating with respect to the fixed stars (Figure 3-2), defined by the axis Ox_i, Oy_i and Oz_i. The Oz_i axis is coincident with Earth's polar axis, i.e. the axis of its rotation.

The Earth fixed frame (**ECEF**, Earth-Centered-Earth-Fixed, e -frame, F_e) has its origin at the centre of the Earth and the axes, which are fixed with respect to the Earth. The axis Oz_e is along the Earth's polar axis; the axis Ox_e lies along the intersection of the plane of the Greenwich meridian with the Earth's equatorial plane and Oy_e is pointed to give a right-handed reference frame. The Earth fix frame rotates with respect to the inertial frame at a rate ω_e about the axis Oz_i (Figure 3-2).

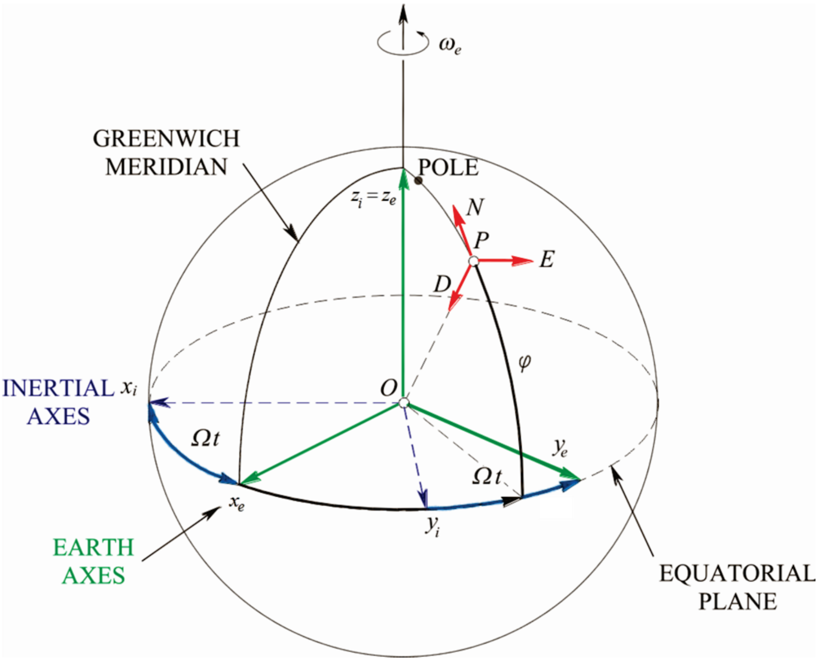


FIGURE 3-2 DEFINITION OF THE INERTIAL AND EARTH FRAMES

The angular rate of the Earth's frame is determined by the matrix column

$$\boldsymbol{\omega}_{ie}^{e} = \begin{bmatrix} 0 & 0 & \Omega \end{bmatrix}^T \tag{3.27}$$

Where Ω is the magnitude of the Earth's angular rate.

The local Earth fixed frame (**NED**, North-East-Down, or **ENUp**, le -frame, F_{le}) has its origin at the missile launch point (start position) on the Earth's surface (or at the initial altitude). The axis Oz_{le} is directed vertically down, and the axis Ox_{le} points North and Oy_{le} East (Figure 3-3).

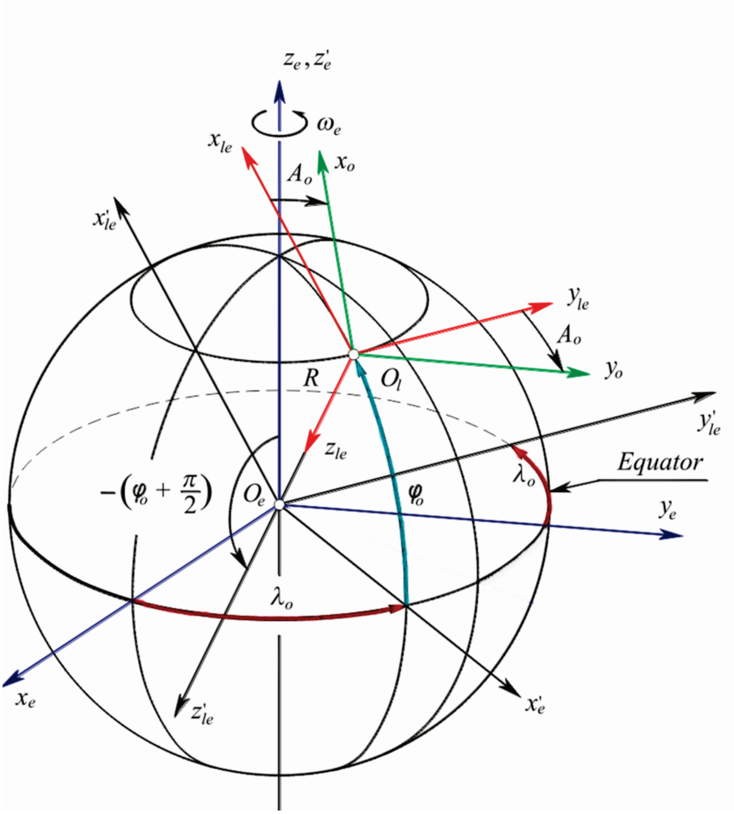


FIGURE 3-3 DEFINITION OF THE LOCAL EARTH FIXED FRAME AND BASIC REFERENCE FRAME

The position of the origin is defined by

$$R_0 + h_0,\ \lambda_0,\ \varphi_0$$

where:

h_0	- the height of the missile at launch point;
R_0	- the medium radius of the Earth;
λ_0	- the longitude of the launch point;
φ_0	- the latitude of the launch point.

The transformation from the Earth frame to the local Earth fixed reference frame can be carried out by using two successive rotations about two axes (Figure 3-3):

- rotate through an angle λ_0 about Oz_e axis and
- rotate through an angle $-\left(\varphi_0 + \dfrac{\pi}{2}\right)$ about the new y_e axis

The transformation matrix from F_e to F_{le} is given by

$$\mathbf{C}_e^{le} = \mathbf{C}_2\left[-\left(\varphi_0 + \frac{\pi}{2}\right)\right]\mathbf{C}_1\left(\lambda_0\right) \tag{3.28}$$

or, having in mind the definition of the matrices $\mathbf{C}_1$ and $\mathbf{C}_2$:

$$\mathbf{C}_e^{le} = \begin{bmatrix} -c\lambda_0 s\varphi_0 & -s\lambda_0 s\varphi_0 & c\varphi_0 \\ -s\lambda_0 & c\lambda_0 & 0 \\ -c\lambda_0 C\varphi_0 & -s\lambda_0 c\varphi_0 & -s\varphi_0 \end{bmatrix} \tag{3.29}$$

(In order to conserve space, the terms "sine" and "cosine" are abbreviated with s and c "respectively")

The angular velocity of the $\mathbf{F}_{le}$ frame is equal to the Earth's rate ω_e. Thus, by using matrix notation, one can obtain:

$$\boldsymbol{\omega}_{ie}^{le} = \mathbf{C}_e^{le}\boldsymbol{\omega}_{ie}^{e} \tag{3.30}$$

Where $\mathbf{C}_e^{le}$ is defined by (3.29) and $\boldsymbol{\omega}_{ie}^{e}$ by (3.27). Substituting (3.29) and (3.27) into Equation (3.30) gives

$$\boldsymbol{\omega}_{ie}^{le} = \begin{bmatrix} \Omega\cos\varphi_0 \\ 0 \\ -\Omega\sin\varphi_0 \end{bmatrix} \tag{3.31}$$

`[lat lon] = dcm2latlon(n)`	*Convert direction cosine matrix between ECEF and NED to geodetic latitude and longitude*
`dcmecef2ned(lat, lon)`	*Convert geodetic latitude and longitude to direction cosine matrix between ECEF and NED*
`dcmeci2ecef(reduction,utc)`	*Convert Earth-centered inertial (ECI) to Earth-centered Earth-fixed (ECEF) coordinates*
`ecef2lla(p)`	*Convert Earth-centered Earth-fixed (ECEF) coordinates to geodetic coordinates*
`eci2lla(position,utc)`	*Convert Earth-centered inertial (ECI) coordinates to latitude, longitude, altitude (LLA) geodetic coordinates*
`lla2ecef(lla)`	*Convert geodetic coordinates to Earth-centered Earth-fixed (ECEF) coordinates*
`lla2eci(lla,utc)`	*Convert geodetic latitude, longitude, altitude (LLA) coordinates to Earth-centered inertial (ECI) coordinates*

TABLE 3-2 AEROSPACE TOOLBOX FUNCTIONS AXIS TRANSFORMATIONS

The basic reference frame (0-frame, F_0) is also Earth fixed frame, but with Ox_0 axis directed to the target or estimated future position of the target. The position of F_0 is determined by the azimuth angle A_0 with respect to the local Earth fixed reference frame F_{le} (Figure 3-3). The origin of the frame F_0 coincides with the launch point. The plane Ox_0y_0 is tangent to the Earth's surface, and the axis Oz_0 is directed vertically down (along the gravity vector).

The transformation from the local Earth fixed reference frame F_{ie} to the basic reference frame F_0 can be carried out through one rotation about the axis Oz_{le} for the azimuth angle A_0. The corresponding transformation matrix is given by

$$\mathbf{C}_{le}^{0} = \mathbf{C}_1\left(A_0\right) = \mathbf{C}\left(A_0, 0, 0\right) \tag{3.32}$$

Having in mind (3.29) and (3.32) the transformation matrix from the Earth reference frame to the basic reference frame is given by the following Equation:

$$\mathbf{C}_{e}^{0} = \mathbf{C}_{le}^{0}\mathbf{C}_{e}^{le} = \mathbf{C}\left(A_0, 0, 0\right)\mathbf{C}_{e}^{le} \tag{3.33}$$

The angular velocity of the basic reference frame is equal to the Earth rate $\boldsymbol{\omega}_e$. By using matrix notation, one can obtain:

$$\boldsymbol{\omega}_{ie}^{0} = \mathbf{C}\left(A_0, 0, 0\right)\boldsymbol{\omega}_{ie}^{le} \tag{3.34}$$

The local geographic (geodetic) reference frame (**NED**, l-frame, F_l) has its origin at the location of the vehicle, point $O \equiv P$, and axes aligned with the directions of North, East and local vertical (down) (Figure 3-4).

The axis Ox_l is chosen to point to the North (N), Oz_l is directed vertically downward (D), i.e., along the local gravity vector $\boldsymbol{g}$ and Oy_l to the East (E). The location of the origin of F_l is determined by $R_0 + h,\ \ \lambda,\ \ \varphi$

where:

h the height of the missile
φ the latitude and
λ the longitude.

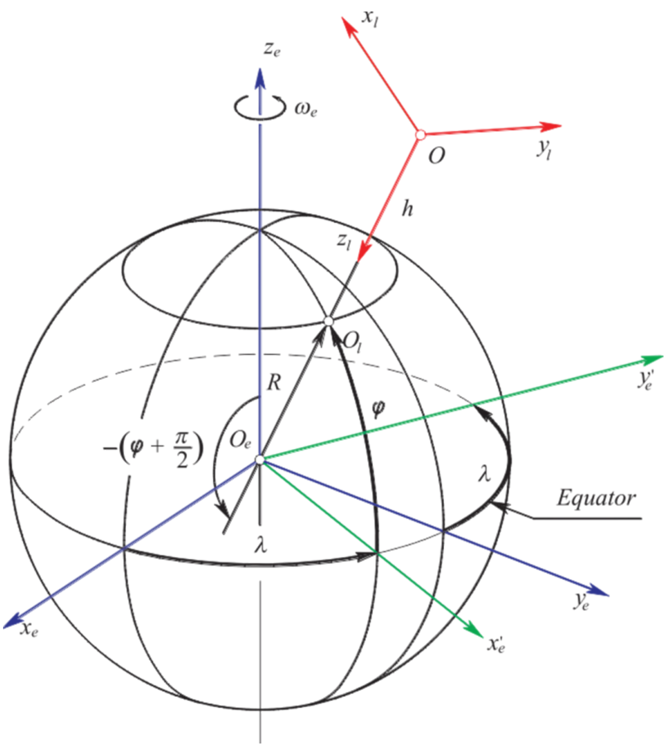


FIGURE 3-4 DEFINITION OF LOCAL GEOGRAPHIC (GEODETIC) REFERENCE FRAME

The transformation matrix from the Earth frame to the local geographic frame can be obtained in a similar way as it was done for local Earth fixed reference frame Equation (3.29). Instead of φ_0 we have φ, and instead of λ_0 we substitute λ.

$$\mathbf{C}_e^l = \begin{bmatrix} -c\lambda s\varphi & -s\lambda s\varphi & c\varphi \\ -s\lambda & c\lambda & 0 \\ -c\lambda c\varphi & -s\lambda c\varphi & -s\varphi \end{bmatrix} \tag{3.35}$$

The angular velocity of the local reference frame, F_l, with respect to the inertial reference frame, F_i, is given by

$$\boldsymbol{\omega}_l = \boldsymbol{\omega}_e + \dot{\boldsymbol{\lambda}} + \dot{\boldsymbol{\varphi}} \tag{3.36}$$

or, in matrix frame:

$$\boldsymbol{\omega}_{il}^l = \mathbf{C}_e^l \boldsymbol{\omega}_{il}^e \tag{3.37}$$

where $\boldsymbol{\omega}_{il}^e$ can be found from Figure 3-4

$$\boldsymbol{\omega}_{il}^{e} = \begin{bmatrix} \dot{\varphi}\sin\lambda \\ -\dot{\varphi}\cos\lambda \\ \Omega + \dot{\lambda} \end{bmatrix} \tag{3.38}$$

Substituting (3.35) and (3.38) into (3.37) one can obtain

$$\boldsymbol{\omega}_{il}^{l} = \begin{bmatrix} \left(\Omega + \dot{\lambda}\right)\cos\varphi \\ -\dot{\varphi} \\ -\left(\Omega + \dot{\lambda}\right)\sin\varphi \end{bmatrix} \tag{3.39}$$

The angular velocity of the local reference frame can be presented as the sum of two components:

- the angular velocity of the Earth with respect to the inertial frame and
- the angular velocity of the local frame with respect to the Earth fixed frame (relative angular velocity)

$$\boldsymbol{\omega}_{il}^{l} = \boldsymbol{\omega}_{ie}^{l} + \boldsymbol{\omega}_{el}^{l} \tag{3.40}$$

where

$$\boldsymbol{\omega}_{ie}^{l} = \left[\Omega\cos\varphi \quad 0 \quad -\Omega\sin\varphi\right]^{T} \tag{3.41}$$

$$\boldsymbol{\omega}_{el}^{l} = \left[\dot{\lambda}\cos\varphi \quad -\dot{\varphi} \quad -\dot{\lambda}\sin\varphi\right]^{T} \tag{3.42}$$

The rate of the local frame with respect to the Earth fixed frame is governed by the motion of the point P with respect to the Earth. If the components of the velocity of the point P relative to the Earth are (Figure 3-4)

v_N - North velocity

v_E - East velocity

v_D - down velocity

the following relations can be written:

$$\dot{\lambda} = \frac{v_E}{\left(R_0 + h\right)\cos\varphi} \tag{3.43}$$

$$\dot{\varphi} = \frac{v_N}{R_0 + h} \tag{3.44}$$

$$\boldsymbol{\omega}_{el}^{l} = \left[\frac{v_E}{R_0 + h} \quad \frac{-v_N}{R_0 + h} \quad \frac{-v_E\tan\varphi}{R_0 + h}\right] \tag{3.45}$$

where R_0 is the radius of the Earth, and h is the height above the surface of the Earth.

The cylindrical reference frame (c -frame, F_c) may be used if the vehicle is to follow a great circle course over the Earth. Like the local geographic reference frame F_l, it is locally leveled, but it is fixed to the vertical plane of the basic reference frame. The relative position of the cylindrical frame and the basic frame is shown in Figure 3-5

The plane of the great circle is determined by the locations of the launch point and the target. (This plane is identical to the vertical plane of the basic reference plane). The origin of the cylindrical frame is defined by the range angle σ and the height of the vehicle above the Earth's surface (h). The axes Ox_c and Oz_c are in the plane of the great circle: Ox_c is directed to the target and Oz_c points down (along with local gravity). The axis Oy_c gives a right-handed reference frame and with Ox_c axis makes the horizontal plane. The departure of the missile from the course is presented by linear quantity y . By using the basic reference frame, the position of the missile may be presented by one curvilinear coordinate ($x = R\sigma$) and two Cartesian coordinates (height h and departure from the course y).

The transformation from the basic reference frame F_0 to the cylindrical frame F_c is carried out through the rotation about the axis Oy_c for the range angle σ (Figure 3-5).

$$\mathbf{C}_0^c = \mathbf{C}_2(-\sigma) = \begin{bmatrix} \cos\sigma & 0 & \sin\sigma \\ 0 & 1 & 0 \\ -\sin\sigma & 0 & \cos\sigma \end{bmatrix} \tag{3.46}$$

or by using the definition of the matrix $\mathbf{C}$:

$$\mathbf{C}_0^c = \mathbf{C}(0, -\sigma, 0) \tag{3.47}$$

The angular velocity of the cylindrical frame with respect to the basic frame is given as

$$\boldsymbol{\omega}_{0c}^c = \begin{bmatrix} 0 & -\dot{\sigma} & 0 \end{bmatrix}^T \tag{3.48}$$

The angular rate of the Earth given with elements in the cylindrical frame is:

$$\boldsymbol{\omega}_{ie}^c = \mathbf{C}_2(-\sigma)\boldsymbol{\omega}_{ie}^0 = \mathbf{C}_2(-\sigma)\mathbf{C}_1(A_0)\boldsymbol{\omega}_{ie}^{le} \tag{3.49}$$

Having in mind the definition of the matrix, $\mathbf{C}$ the Equation (3.49) may be transformed into:

$$\boldsymbol{\omega}_{ie}^c = \mathbf{C}(A_0, -\sigma, 0)\boldsymbol{\omega}_{ie}^{le} \tag{3.50}$$

where the angular rate of Earth with elements in the Earth fixed frame $\boldsymbol{\omega}_{ie}^{le}$ is given by Equation (3.31)

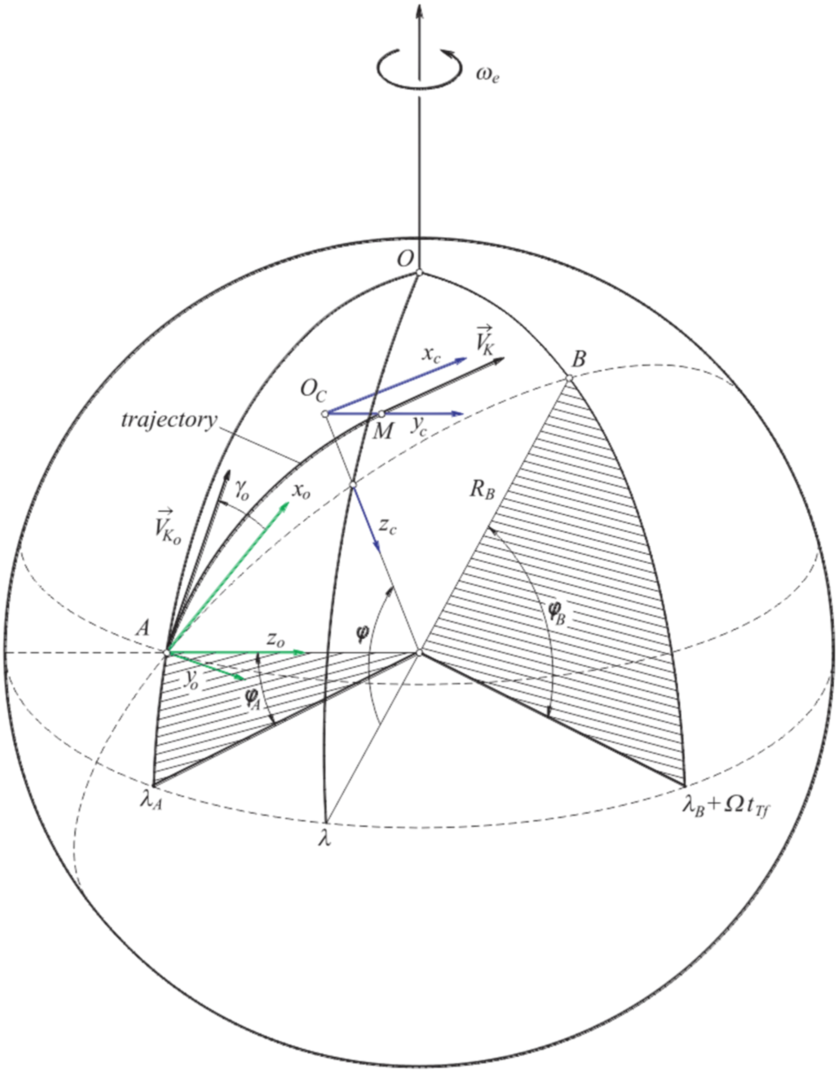


FIGURE 3-5 CYLINDRICAL REFERENCE FRAME

The angular rate of the cylindrical frame with respect to the inertial frame may be developed having in mind

$$\boldsymbol{\omega}_c = \boldsymbol{\omega}_e + \dot{\boldsymbol{\sigma}} \tag{3.51}$$

or, in the matrix form

$$\boldsymbol{\omega}_{ic}^c = \boldsymbol{\omega}_{ie}^c + \boldsymbol{\omega}_{0c}^c \tag{3.52}$$

where $\boldsymbol{\omega}_{ie}^c$ and $\boldsymbol{\omega}_{0c}^c$ are computed by (3.49) and (3.48) respectively.

The turn rate of the cylindrical frame with respect to the Earth, $\boldsymbol{\omega}_{0c}^c$, is governed by the motion of its origin with respect to the Earth. If the component of the velocity with respect to the Earth along x_c axis is noted by v_{x_c} one can obtain:

$$\dot{\sigma} = \frac{v_{x_c}}{R_0 + h} \tag{3.53}$$

where h is the height of the missile above Earth's surface.

The missile–carried vertical frame (v -frame, F_v) has its origin at the mass centre. Depending on the problem which should be solved, the axes of the F_v frame are parallel to the corresponding axes of the following reference frames

- local geographic reference frame, F_l,
- basic reference frame, F_0, and
- cylindrical reference frame, F_c.

The angular orientation of the reference frame F_v and its angular velocity are identical to these quantities of one of the above reference frames.

The body-fixed reference frame (b -frame, F_b) is the system attached to the body of the missile (Figure 3-6).

The origin of F_b is the mass centre $(O \equiv CM)$. Ox is the longitudinal missile axis through the fuselage toward the nose. Oz is in the plane of symmetry and is directed downward Oy gives a right-handed reference frame.

The position of the body frame F_b with respect to the missile carried vertical reference frame F_v is determined by three angles (Figure 3-6) which are referred to as the Euler rotation angles.

Three successive rotations are as follows:

- rotation about Oz_v axis for the yaw angle ψ
- rotation about a new y axis for pitch angle θ and
- rotation about x axis for the roll angle ϕ.

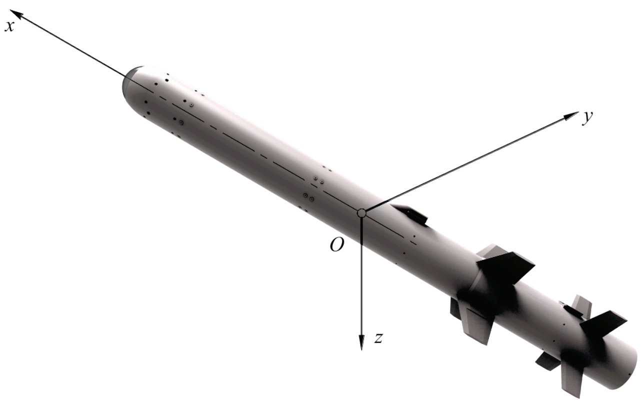


FIGURE 3-6 BODY REFERENCE FRAME

The transformation matrix from F_v to F_b may be expressed as the product of these three separate transformations as follows

$$\mathbf{C}_v^b = \mathbf{C}_3(\phi)\mathbf{C}_2(\theta)\mathbf{C}_1(\psi) = \mathbf{C}(\psi,\theta,\phi) \tag{3.54}$$

where $\mathbf{C}_1$, $\mathbf{C}_2$, $\mathbf{C}_3$ and $\mathbf{C}$ are defined in Chapter 3.2 Notation and Transformation.

The angular velocity of the body frame with respect to the inertial frame F_i is given in terms of three components along its axis:

$$\boldsymbol{\omega}_{ib}^b = [p \quad q \quad r]^T \tag{3.55}$$

where:

- p is the roll rate (along Ox)
- q is the pitch rate (along Oy) and
- r is the yaw rate (along Oz)

The body frame has an essential role in flight mechanics. The subscript or superscript may be dropped when there is no confusion

The angular velocity of the body frame with respect to the missile–carried vertical frame can be found in Figure 3-1.

$$\boldsymbol{\omega}_r = \dot{\boldsymbol{\psi}} + \dot{\boldsymbol{\theta}} + \dot{\boldsymbol{\phi}} \tag{3.56}$$

or in the matrix notation

$$\boldsymbol{\omega}_{vb}^{b} = \begin{bmatrix} p_r \\ q_r \\ r_r \end{bmatrix} = \begin{bmatrix} \dot{\phi} \\ 0 \\ 0 \end{bmatrix} + \mathbf{C}_3(\phi)\begin{bmatrix} 0 \\ \dot{\theta} \\ 0 \end{bmatrix} + \mathbf{C}_3(\phi)\mathbf{C}_2(\theta)\begin{bmatrix} 0 \\ 0 \\ \dot{\psi} \end{bmatrix} \tag{3.57}$$

This Equation can be arranged and expressed in component form as:

$$\begin{aligned} \dot{\phi} &= (q_r \sin\phi + r_r \cos\phi)\tan\theta + p_r \\ \dot{\theta} &= q_r \cos\phi - r_r \sin\phi \\ \dot{\psi} &= (q_r \sin\phi + r_r \cos\phi)\sec\theta \end{aligned} \tag{3.58}$$

The aeroballistic reference frame (ab-frame, F_{ab}) is the system attached to the body, but it does not rotate about the longitudinal axis. The $O\tilde{z}$ axis (Figure 3-1) is always in the vertical plane and $O\tilde{y}$ in the horizontal plane.

Assuming the missile-carried vertical frame is the inertial references frame, we write components of the missile angular rate with respect to the inertial frame (p, q, r) instead of (p_r, q_r, r_r):

$$\begin{aligned} \dot{\phi} &= (q \sin\phi + r \cos\phi)\tan\theta + p \\ \dot{\theta} &= q \cos\phi - r \sin\phi \\ \dot{\psi} &= (q \sin\phi + r \cos\phi)\sec\theta \end{aligned} \tag{3.59}$$

The relationships between the angular rate of the aeroballistic frame and the Euler's angles are obtained from (3.59) for $\phi = 0$ and $\dot{\phi} = 0$:

$$\begin{aligned} 0 &= \tilde{r} \tan\theta + p_{ab} \\ \dot{\theta} &= \tilde{q} \\ \dot{\psi} &= \tilde{r} \sec\theta = \frac{\tilde{r}}{\cos\theta} \end{aligned} \tag{3.60}$$

The angular rate of the aeroballistic frame about the longitudinal missile axis is:

$$p_{ab} = -\tilde{r} \tan\theta = -\dot{\psi} \sin\theta \tag{3.61}$$

Hence the angular rate of the aeroballistic reference frame with respect to the inertial frame is:

$$\boldsymbol{\omega}_{i,ab}^{ab} = \begin{bmatrix} p_{ab} \\ \tilde{q} \\ \tilde{r} \end{bmatrix} = \begin{bmatrix} -\dot{\psi}\sin\theta \\ \dot{\theta} \\ \dot{\psi}\cos\theta \end{bmatrix} \tag{3.62}$$

For analysis purposes, the segment of trajectory with the small value of the slope the angular rate p_{ab} can be neglected, and we obtain:

$$\boldsymbol{\omega}_{i,ab}^{ab} = \begin{bmatrix} 0 \\ \tilde{q} \\ \tilde{r} \end{bmatrix} = \begin{bmatrix} 0 \\ \dot{\theta} \\ \dot{\psi}\cos\theta \end{bmatrix} \tag{3.63}$$

The angular velocity of the missile body with components in the aeroballistic frame is:

$$\boldsymbol{\omega}_{i,b}^{ab} = \boldsymbol{\omega}^{ab} = \begin{bmatrix} p \\ \tilde{q} \\ \tilde{r} \end{bmatrix} = \begin{bmatrix} p_{ab} + \dot{\phi} \\ \dot{\theta} \\ \dot{\psi}\cos\theta \end{bmatrix} \approx \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi}\cos\theta \end{bmatrix} \tag{3.64}$$

and expressed in terms of the components in the body reference frame

$$\boldsymbol{\omega}_{i,b}^{ab} = \boldsymbol{\omega}^{ab} = \begin{bmatrix} p \\ \tilde{q} \\ \tilde{r} \end{bmatrix} = \mathbf{C}_b^{ab}\boldsymbol{\omega}^b = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} p \\ q\cos\phi - r\sin\phi \\ q\sin\phi + r\cos\phi \end{bmatrix} \tag{3.65}$$

or

$$\begin{aligned} \tilde{q} &= q\cos\phi - r\sin\phi \\ \tilde{r} &= q\sin\phi + r\cos\phi \end{aligned} \tag{3.66}$$

The aerodynamic reference frame (a-frame, F_a) is attached to the body and can be obtained from the body reference frame when the $\overline{y}$ and $\overline{z}$ axes do not roll with the body, but $O\overline{z}$ axis remains in the plane of incidence. The aerodynamic reference frame $\left(O\overline{x}\,\overline{y}\,\overline{z}\right)$ is determined with aerodynamic roll angle ϕ_a with respect to the body frame (Figure 3-7).

The aerodynamic frame is chosen to represent the aerodynamic force and moment data as the input for the simulation of flight.

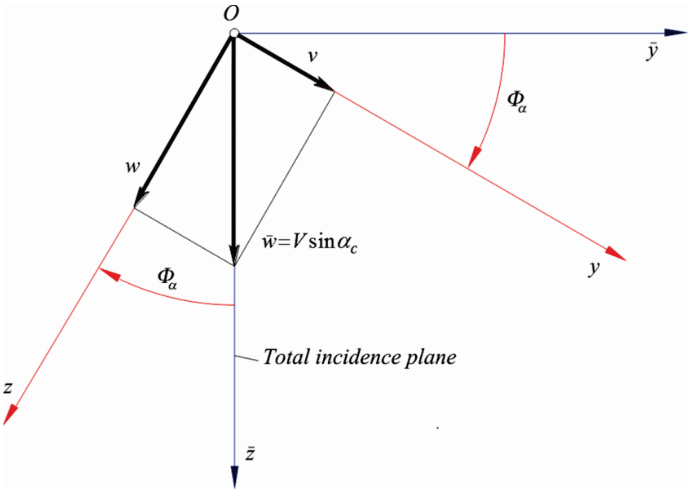


FIGURE 3-7 AERODYNAMIC REFERENCE FRAME

3.5 Transformations defined by Quaternions

Quaternion algebra is one possible way to describe the 3D orientation and rotation of a rigid body. They mathematically represent a non-commutative form of complex numbers. Quaternions were first described by the Irish mathematician Sir William Rowan Hamilton in 1843 and applied to the mechanics of three-dimensional space. Today, they are used in both theoretical and applied mathematics, especially for calculations of rotations in three-dimensional space. Quaternions form a 4-dimensional normed algebra over real numbers, in which each object consists of 4 scalar quantities (sometimes called Euler parameters, which is not the same as Euler angles). Quaternion algebra is usually denoted by $\mathbf{H}$ (by Hamilton)

As previously mentioned, quaternions are 4D objects, which consist of one real part and three imaginary parts. Each of these imaginary dimensions has a unit value of the square root of -1, but they do not represent the same size of the square root of -1, but are mutually normal to each other and are denoted by i, j, k. Thus, we can now define the quaternion as a unique linear combination of its elements as follows:

$$q = a + bi + cj + dk \tag{3.67}$$

3.5.1 Rotation defined by Quaternions

Suppose we represent the final rotation by the rotation about a fixed axis for a certain angle where the axis unit vector is $\left(e_x, e_y, e_z\right)$ and the angle θ. In that case, the equivalent quaternion can be represented as follows:

$$a = \cos\frac{\theta}{2}$$
$$\mathbf{e} = \begin{bmatrix} b \\ c \\ d \end{bmatrix} = \vec{n}\sin\frac{\theta}{2} = \left(e_x, e_y, e_z\right)^T \sin\frac{\theta}{2} \tag{3.68}$$
$$q = \left(a, \mathbf{e}\right) = a + bi + cj + dk$$

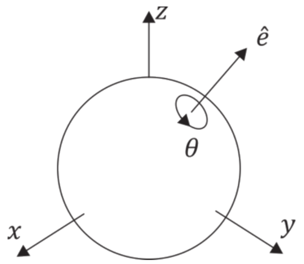


FIGURE 3-8 ROTATION REPRESENTED BY AXIS ANGLE NOTATION

Although rotation in 3D has three degrees of freedom, it does not represent 3D vector space. Moreover, the representation of rotation in 3D using three scalar quantities is not linear and has singularities (Euler angles). To overcome these problems, a 4D quaternion space is used to describe 3D rotation.

Probably the essential application of quaternions is to represent rotation in 3D space using them. The formula for 3D rotation using quaternions is as follows:

$$p' = qpconj\left(q\right) \tag{3.69}$$

where:

p', p - points in 3D space represented by their coordinates

$p = xi + yj + zk \rightarrow p = \left(0, x, y, z\right)$

$q, conj(q)$ - Quaternion and quaternion conjugate

3.5.2 Quaternions algebra

3.5.2.1 Addition (subtraction) of quaternions

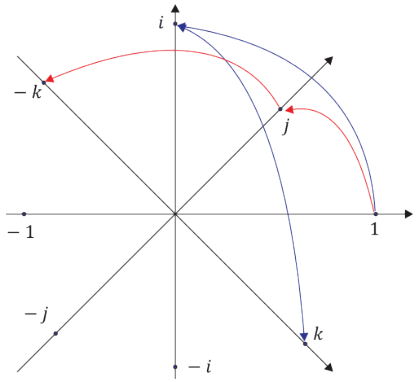
Addition (subtraction) of quaternions is done by independently adding (subtracting) coordinates (as in the addition of vectors or complex numbers).

$$q_1 = a_1 + b_1 i + c_1 j + d_1 k$$
$$q_2 = a_2 + b_2 i + c_2 j + d_2 k \tag{3.70}$$
$$q_3 = q_1 \pm q_2 = (a_1 \pm a_2) + (b_1 \pm b_2) i + (c_1 \pm c_2) j + (d_1 \pm d_2) k$$

3.5.2.2 Quaternions multiplication. Hamilton's product.

The multiplication of quaternions can be compared to the multiplication of complex numbers with the following rules:

$$ii = jj = kk = ijk = -1$$
$$ij = k \quad ji = -k$$
$$jk = i \quad kj = -i$$
$$ki = j \quad ik = -j$$



From the previously mentioned rules, it can be concluded that commutativity does not apply to quaternion multiplication and that the order of multiplication is important. Physically, this property of quaternions can be explained by rotation (for the representation of which quaternions are used), so if we rotate 90 degrees relative to the x axis and then 90 degrees relative to the y axis, we get a different solution if we first rotate about the y axis 90 degrees and then around the x axis by 90 degrees.

In the general case, the multiplication of quaternions is neither commutative nor anti-commutative because it also contains commutative parts (multiplication of a scalar part by a scalar, multiplication of a scalar part by imaginary, multiplication of imaginary parts with the same imaginary unit).

$$q_1 = a_1 + b_1 i + c_1 j + d_1 k$$
$$q_2 = a_2 + b_2 i + c_2 j + d_2 k$$
$$q_3 = q_1 \cdot q_2 = a_3 + b_3 i + c_3 j + d_3 k$$
$$a_3 = a_1 a_2 - b_1 b_2 - c_1 c_2 - d_1 d_2 \tag{3.71}$$
$$b_3 = a_1 b_2 + b_1 a_2 + c_1 d_2 - d_1 c_2$$
$$c_3 = a_1 c_2 - b_1 d_2 + c_1 a_2 + d_1 b_2$$
$$d_3 = a_1 d_2 + b_1 c_2 - c_1 b_2 + d_1 a_2$$

3.5.2.3 CONJUGATION. NORM. QUATERNION DIVISION

The conjugation of quaternions is analogue to the conjugation of complex numbers. So if we have a quaternion, $q = a + bi + cj + dk$ then quaternion conjugate is:

$$conj(q) = a - bi - cj - dk \tag{3.72}$$

And the following features of the conjugation operation apply:

$$\begin{aligned} &conj(conj(q)) = q \\ &conj(q_1 q_2) = conj(q_2) conj(q_1) \end{aligned} \tag{3.73}$$

The norm (intensity) of a quaternion is the square root of a quaternion and its conjugate:

$$\|q\| = \sqrt{q\, conj(q)} = \sqrt{a^2 + b^2 + c^2 + d^2} \tag{3.74}$$

A unit quaternion is a quaternion whose norm is equal to 1; it is also called the normalized or versor:

$$Uq = \frac{q}{\|q\|} \tag{3.75}$$

Due to the non-commutative multiplication of the quaternion, the division will be represented as multiplication with the inverse quaternion.

$$\begin{aligned} &q^{-1} = \frac{conj(q)}{\|q\|^2} = \frac{conj(a + bi + cj + dk)}{a^2 + b^2 + c^2 + d^2} = \frac{a - bi - cj - dk}{a^2 + b^2 + c^2 + d^2} \\ &q\, conj(q) = \|q\|^2 \end{aligned} \tag{3.76}$$

3.5.3 RIGID BODY ROTATION. RELATIONSHIP BETWEEN THE ANGULAR VELOCITY OF ROTATION AND QUATERNION

Suppose that a rigid body rotates about x, y, and z axis simultaneously, with angular velocities p, q, r, respectively. Then, the equivalent rotation about the axis can represent this rotation:

$$\boldsymbol{u} = \frac{1}{|\omega|}(p\boldsymbol{i} + q\boldsymbol{j} + r\boldsymbol{k}) \tag{3.77}$$

where $|\omega| = \sqrt{p^2 + q^2 + r^2}$

Now, for the time interval, dt the angle will change for $|\omega(t)|dt$, which we can express via quaternion e:

$$e = \left(\cos\left(\frac{|\omega|dt}{2}\right), u\sin\left(\frac{|\omega|dt}{2}\right)\right)$$
$$e = \cos\left(\frac{|\omega|dt}{2}\right) + i\sin\left(\frac{|\omega|dt}{2}\right)\frac{p}{|\omega|} + j\sin\left(\frac{|\omega|dt}{2}\right)\frac{q}{|\omega|} + k\sin\left(\frac{|\omega|dt}{2}\right)\frac{r}{|\omega|} \tag{3.78}$$

Now we need to calculate the first derivative of the quaternion by time:

$$\frac{de}{dt} = \frac{d}{dt}\left(\cos\left(\frac{|\omega|dt}{2}\right)\right) + i\frac{d}{dt}\left(\sin\left(\frac{|\omega|dt}{2}\right)\frac{p}{|\omega|}\right) +$$
$$+ j\frac{d}{dt}\left(\sin\left(\frac{|\omega|dt}{2}\right)\frac{q}{|\omega|}\right) + k\frac{d}{dt}\left(\sin\left(\frac{|\omega|dt}{2}\right)\frac{r}{|\omega|}\right) \tag{3.79}$$

or:

$$\frac{de}{dt} = \frac{1}{2}\left(-\sin\left(\frac{|\omega|dt}{2}\right)|\omega| + i\cos\left(\frac{|\omega|dt}{2}\right)p + j\cos\left(\frac{|\omega|dt}{2}\right)q + k\cos\left(\frac{|\omega|dt}{2}\right)r\right) \tag{3.80}$$

We can now modify the previously obtained expression into the following:

$$\frac{de}{dt} = \frac{1}{2}\left(-\sin\left(\frac{|\omega|dt}{2}\right)|\omega|\left(\frac{p^2}{|\omega|^2} + \frac{q^2}{|\omega|^2} + \frac{r^2}{|\omega|^2}\right)\right) +$$
$$+\frac{1}{2}i\left(p\cos\left(\frac{|\omega|dt}{2}\right) + q\sin\left(\frac{|\omega|dt}{2}\right)\frac{r}{|\omega|} - r\sin\left(\frac{|\omega|dt}{2}\right)\frac{q}{|\omega|}\right) +$$
$$+\frac{1}{2}j\left(q\cos\left(\frac{|\omega|dt}{2}\right) + r\sin\left(\frac{|\omega|dt}{2}\right)\frac{p}{|\omega|} - p\sin\left(\frac{|\omega|dt}{2}\right)r\right) +$$
$$+\frac{1}{2}k\left(r\cos\left(\frac{|\omega|dt}{2}\right) + p\sin\left(\frac{|\omega|dt}{2}\right)\frac{q}{|\omega|} - q\sin\left(\frac{|\omega|dt}{2}\right)\frac{p}{|\omega|}\right) \tag{3.81}$$

Which is equivalent to the following expression:

$$\frac{de}{dt}=\frac{1}{2}\left(-p\sin\left(\frac{|\omega|dt}{2}\right)\frac{p}{|\omega|}-q\sin\left(\frac{|\omega|dt}{2}\right)\frac{q}{|\omega|}-r\sin\left(\frac{|\omega|dt}{2}\right)\frac{r}{|\omega|}\right)+$$
$$+\frac{1}{2}i\left(p\cos\left(\frac{|\omega|dt}{2}\right)+q\sin\left(\frac{|\omega|dt}{2}\right)\frac{r}{|\omega|}-r\sin\left(\frac{|\omega|dt}{2}\right)\frac{q}{|\omega|}\right)+$$
$$+\frac{1}{2}j\left(-p\sin\left(\frac{|\omega|dt}{2}\right)\frac{r}{|\omega|}+q\cos\left(\frac{|\omega|dt}{2}\right)+r\sin\left(\frac{|\omega|dt}{2}\right)\frac{p}{|\omega|}\right)+ \quad (3.82)$$
$$+\frac{1}{2}k\left(p\sin\left(\frac{|\omega|dt}{2}\right)\frac{q}{|\omega|}-q\sin\left(\frac{|\omega|dt}{2}\right)\frac{p}{|\omega|}+r\cos\left(\frac{|\omega|dt}{2}\right)\right)$$

Based on the previously arranged expression, we can conclude:

$$\frac{de}{dt}=\frac{1}{2}(0,p,q,r)\left(\cos\left(\frac{|\omega|dt}{2}\right),\sin\left(\frac{|\omega|dt}{2}\right)\frac{p}{|\omega|},\sin\left(\frac{|\omega|dt}{2}\right)\frac{q}{|\omega|},\sin\left(\frac{|\omega|dt}{2}\right)\frac{r}{|\omega|}\right) \quad (3.83)$$
$$\frac{de}{dt}=\frac{1}{2}\omega(t)q(t)$$

The Equation (3.83) is equivalent to the Equation (3.59) but written in terms of quaternions.

`angle2quat(Ang1,Ang2,Ang3, rotationSequence)`	*Convert rotation angles to quaternion*
`dcm2quat(n)`	*Convert direction cosine matrix to quaternion*
`quat2angle(q)`	*Convert quaternion to rotation angles*
`quat2dcm(q)`	*Convert quaternion to direction cosine matrix*
`quatconj(q)`	*Calculate conjugate of quaternion*
`quatmultiply(q,r)`	*Calculate product of two quaternions*
`quatnorm(q)`	*Calculate norm of quaternion*
`quatnormalize(q)`	*Normalize quaternion*
`quatrotate(q,r)`	*Rotate vector r by quaternion q*

TABLE 3-3 AEROSPACE TOOLBOX, QUATERNION ALGEBRA AND TRANSFORMATIONS VIA QUATERNIONS

4 Aerodynamic Forces & Moments

4.1 BASIC CONCEPTS AND DEFINITIONS

The aerodynamic forces and moments will be briefly discussed in this section. We will assume a typical skid-to-turn missile of cruciform configuration (Figure 4-1). Missiles, particularly interceptor weapons, usually require to be rapidly manoeuvred, which is easy to achieve by the use of a cruciform arrangement of wings. This configuration has a lift capability nearly the same in every radial plane. The layout of aerodynamic controls located at the rear is very common in application. These controls are all moving about their hinge axes. If opposite controls are deflected in pairs in the same direction, they give increments in moments about the pitch or yaw axis. In this case, they are used as pitch controls (elevators) or yaw controls (rudders). If opposite controls are deflected in opposite directions, they will produce a rolling moment but no pitch or yaw. Now, they are used as roll controls (ailerons). The aerodynamic controls are moved in such a way to produce the required combinations of the moment about the three axes.

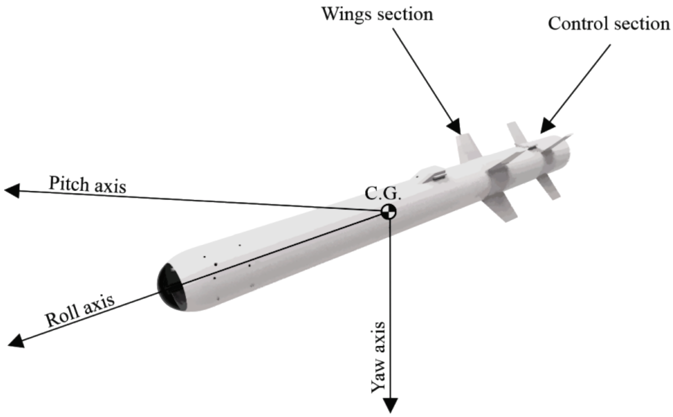


FIGURE 4-1 TYPICAL SKID-TO-TURN CRUCIFORM MISSILE

The magnitudes of forces and moments that act on a missile depend on the combined effects of many different variables. The following parameters govern the magnitude of aerodynamic forces and moments: configuration geometry, angle of attack, missile size, free-stream velocity, the density of undisturbed air, Reynolds number that includes viscous effects, and Mach number that describes compressibility effects. In order to correlate the data for various stream conditions and configurations, the measurements are usually presented in dimensionless form, i.e. they are presented by aerodynamic coefficients. However, in some cases, flow phenomena such as boundary-layer separation, shock-wave/boundary-layer interaction and compressibility effects may limit the range of flow conditions over which the aerodynamic coefficients remain applicable in practice. In addition, the motion of air around a missile produces pressure and velocity variations, which produce the aerodynamic forces and moments.

The forces acting on a missile in flight consist of aerodynamic, propulsive (thrust) that will be described in Chapter 05, and gravitational force (Chapter 02). These forces can be resolved along the missile's body-axis system ($Oxyz$) and fixed to the missile's centre of gravity (cg).

The reference axes system standardised in guided weapons is centred on cg and fixed in the body. Thus, any set of axes fixed in a rigid body is a body-fixed reference frame.

Before we proceed with the present discussion, some of the fundamental concepts and definitions of aerodynamics will be reviewed. These definitions and nomenclature will be given with reference to Figure 4-2, which presents an airfoil in a wind tunnel with free-stream velocity V_∞ .

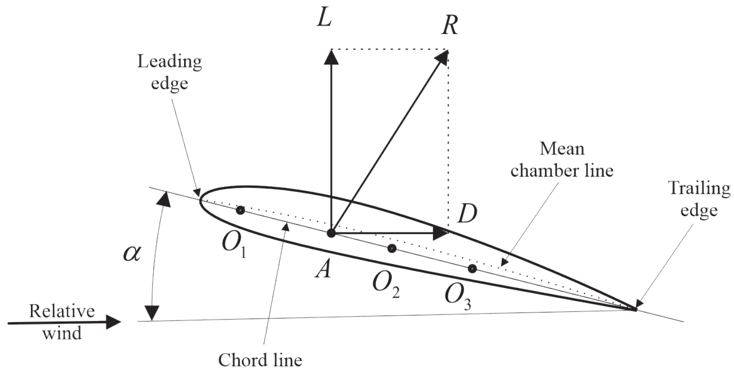


FIGURE 4-2 NOMENCLATURE AND DEFINITIONS ($A \equiv cp$).

The angle of attack (α) is the angle between the relative wind and chord line (or missile's longitudinal axis).

Aerodynamic centre (*ac*)*:* The point on the chord of an airfoil about which the moment coefficient is practically constant for all angles of attack. Since the moment about the aerodynamic centre is the product of the lift that acts at the centre of pressure and a lever arm (the distance from the aerodynamic centre to the centre of pressure), the centre of pressure must move toward the aerodynamic centre as the lift increases.

The centre of pressure (*cp*) is the point on the chord of an airfoil through which all aerodynamic forces act. The centre of pressure, in general, will not be located at the aerodynamic centre of the airfoil.

Dynamic pressure: The aerodynamic pressure frequently appears in the derivation of aerodynamic formulas. Dynamic pressure, denoted by the symbol Q , is given by the expression $Q = \frac{1}{2}\rho V_\infty^2$, where ρ is the air density and V_∞ is the free-stream velocity.

Relative wind: refers to the motion of air relative to an airfoil or a missile and opposite to the forward velocity of the air vehicle in a steady-state atmosphere.

Resultant aerodynamic force is the vector summation of all the aerodynamic forces acting on the airfoil. Its point of application is at the centre of pressure.

Consider now the airfoil in Figure 4-2, which is at an angle of attack in the uniform stream V_∞ of wind tunnel and is pivoted about a point *O* on the chord line. This pivot point is capable of being altered to any of the three positions O_1, O_2, O_3. There is a resultant force $\boldsymbol{R}$ that acts through the centre of pressure, *cp*. This force produces a pitching moment which is positive if it tends to increase α.

The aerodynamic force depends significantly upon the airfoil geometry, which is governed by thickness t/c, the shape of the camber line, and leading-edge radius r_{LE} (Figure 4-3).

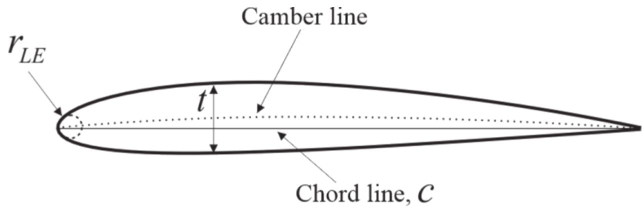


FIGURE 4-3 AIRFOIL GEOMETRY

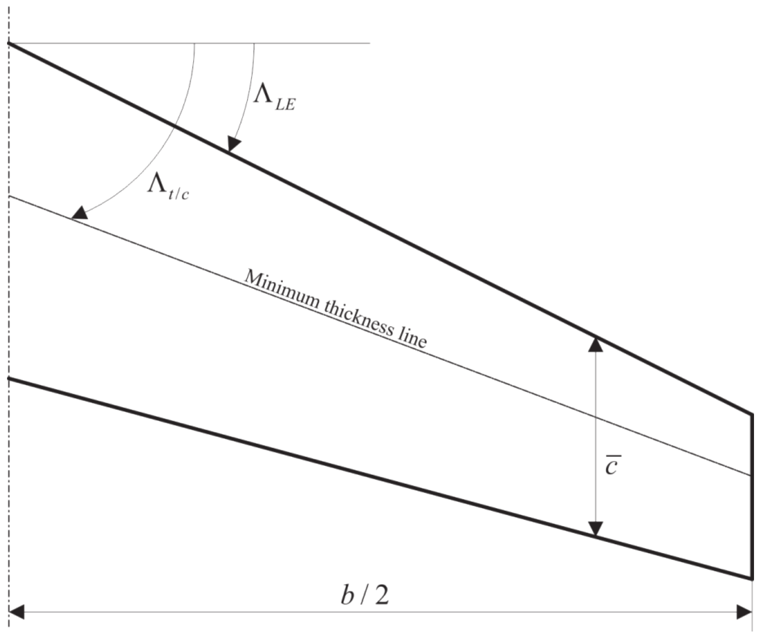


FIGURE 4-4 WING PLANFORM GEOMETRY

There is a reduction in the lift force due to the finite wingspan, *b*, due to the presence of wing-tip vortices. (Wing-tip vortices refer to the rotary flow due to the leakage around wing-tips of high-pressure air from the lower surface to the lower-pressure region of the upper surface. Due to this leakage of flow, there is a loss of lift because of a reduction in the pressure difference between the lower and upper surfaces compared to a wing of finite span.)

A non-dimensional geometric parameter $A = \frac{b^2}{S}$, called the aspect ratio, governs the wing-tip vortices' strength and affects the lift-dependent drag (induced drag). For a span wing, the aerodynamic force is a complex function of the wing planform geometry (apart from the airfoil shape) and depends upon the aspect ratio and the sweep angles $\Lambda_{LE}, \Lambda_{t/c}$ (Figure 4-4).

4.2 Main Aerodynamic Forces and Moments

It is usual in aerodynamics to resolve the sum of the normal (due to pressure) forces and the tangential (due to viscous shear) forces that act on the surface due to the fluid motion around a vehicle into three components along axes parallel and perpendicular to the free-stream direction. These forces are lift (*L*), drag (*D*), and side force (*Y*). The relation of lift and drag forces to the free-stream direction is shown in Figure 4-2 and Figure 4-5.

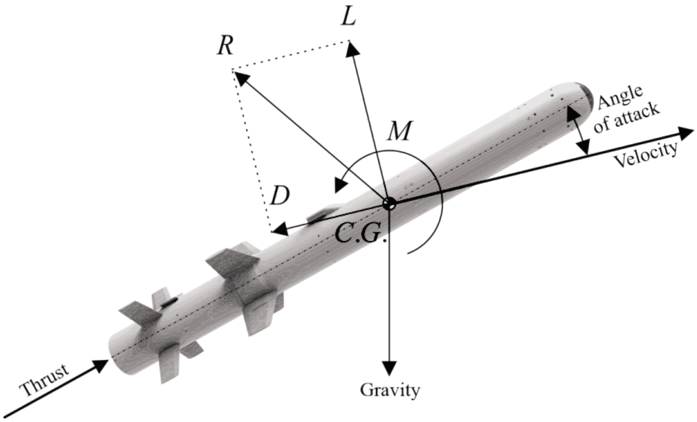


FIGURE 4-5 AERODYNAMIC FORCES, THRUST AND GRAVITY ACTING ON A MISSILE

Lift is the resultant aerodynamic force component that is perpendicular (upward) to the relative wind or to the undisturbed free-stream velocity. This force is produced primarily by the pressure forces acting on the missile surface. Thus, the lift force is perpendicular to the missile's velocity vector in the plane of the angle of attack.

Drag is the resultant aerodynamic force component parallel to the relative wind, or it is a net aerodynamic force in the direction of undisturbed free-stream velocity. The aerodynamic drag is produced by the pressure and skin friction forces on the surface. The drag force is measured along the velocity vector but in the opposite direction.

Side force is the component of force in a direction perpendicular to both the lift and the drag and is measured in the direction normal to the plane of the angle of attack. The side force is positive when acting on the right from the plane of the angle of attack, looking in the direction of the flight of a missile.

The basic aerodynamic forces are commonly defined in terms of dimensionless coefficients, the dynamic pressure and reference area:

$$\text{Drag: } D = C_D QS \tag{4.1}$$

$$\text{Lift: } L = C_L QS \tag{4.2}$$

$$\text{Side force: } Y = C_Y QS \tag{4.3}$$

where

C_D	Drag coefficient,
C_L	Lift coefficient,
C_Y	Side force coefficient,
$S = \dfrac{d^2\pi}{4}$	Reference area

d -is the reference diameter. The reference diameter is usually taken as the diameter of the cylindrical section immediately following the end of the ogive. It is permissible, however, to use any convenient value as the reference diameter as long as the selected dimension is clearly specified and illustrated.

The moment of L and D (Figure 4-2) about O is

$$M = OA(L\cos\alpha + D\sin\alpha) \tag{4.4}$$

Where OA is positive when *cp* is in front of O.

For a well-designed airfoil ($D \ll L$) and small angle of attack, the moment may be approximated by

$$M = OA \times L \tag{4.5}$$

If we define aerodynamic moment by non-dimensional coefficient C_m with a chord as the characteristic length

$$M = C_m QSc \tag{4.6}$$

it follows

$$OA = \frac{M}{L} = \frac{C_m}{C_L} \times c \tag{4.7}$$

Over the linear portion of $C_L(\alpha)$, the associated moment function $C_m(\alpha)$ is also linear:

$$C_m = C_{m0} + B \times C_L \tag{4.8}$$

Where C_{m0} and B are constant.

The intersection with ordinate C_{m0} is the pitching moment coefficient at zero lift and is independent of the point about which C_m is measured (it is a pure couple). The slope $B = \dfrac{dC_m}{dC_L}$ depends on the pivot point O.

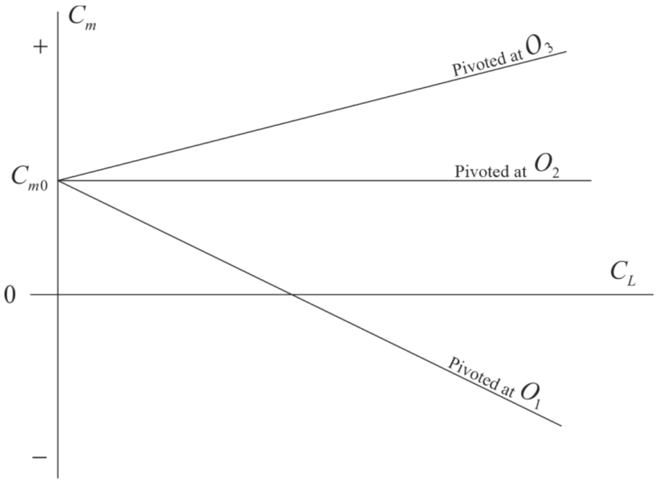


FIGURE 4-6 DIAGRAMS $C_m = f(C_L)$ FOR VARIOUS POSITIONS OF THE PIVOT POINT

Figure 4-6 shows the $C_m(\alpha)$ curves for the points O_1, O_2, O_3. If point O is moved backwards from the forward position O_1 through O_2 to the rear position O_3, there is a change of sign of B from negative to positive. At the position, O_2 it may be

$$\frac{dC_m}{dC_L} = 0 \tag{4.9}$$

This position on the chord is known as the aerodynamic centre (*ac*). From Equation (4.7) we get the change of aerodynamic moment ΔM due to the change of lift ΔL :

$$\Delta M = \frac{dC_m}{dC_L} \Delta C_L Q c = \left(\frac{dC_m}{dC_L} c \right) \Delta L \tag{4.10}$$

It follows that the distance of the aerodynamic centre, *ac* from general pivot point O is

$$x_{ac} = \frac{dC_m}{dC_L} c \tag{4.11}$$

The distance of the centre of pressure from O is obtained from Equations (4.7) and (4.8)

$$x_{cp} = \frac{C_m}{C_L} c = \left(\frac{C_{m0}}{C_L} + \frac{dC_m}{dC_L} \right) c \tag{4.12}$$

Therefore, in the case of $C_{m0} = 0$ we get $x_{ac} = x_{cp}$

The quantity C_{m0} arises from the wing camber, which for most missile wings is zero, and the aerodynamic centre is identical to the centre of pressure. However, cruise missiles may have cambered wings, and it is necessary to make a difference between *ac* and *cp*.

If the pivot point is at the aerodynamic centre, a change of α and C_L produces no change in pitching moment, and this is a state of neutral equilibrium. However, if a pivot point is behind the aerodynamic centre such as O_3, an increase of α and C_L will produce a positive increment of pitching moment which tends to increase α even more. Similarly, a decrease of α and C_L creates a negative increment in C_m which tends to further decrease α. Such arrangement is known as a state of unstable equilibrium. Opposite to this is a pivot point such as O_1 which is in front of the aerodynamic centre. We have that the positive increment of α and C_L produce negative pitching moment which tends to decrease the angle of attack. This is a state of stable equilibrium.

If now we transfer the pivoted aerofoil to a situation of free flight of the missile, the pitching moment induced by a change of angle of attack would act about the centre of gravity (*cg*) instead of a pivot point. Since we ignore the pitching moment due to $\dot{\alpha}$ or $\ddot{\alpha}$ we refer to this as the static stability.

The definitions of the aerodynamic forces, moments and velocity components in the body-fixed reference system, which will be used in the application for missiles, are summarised in Figure 4-7.

The notations in Figure 4-7 are as follows:

x, y, z	distance coordinates with reference to the origin *O*,
U, v, w	velocity components,
X, Y, Z	force components (axial, side and normal force, respectively),
p, q, r	angular velocity components (rates of roll, pitch and yaw, respectively),
L, M, N	components of the moment (roll, pitch and yaw, respectively).

Since the longitudinal component of velocity is usually much greater than perpendicular ones, we use the capital letter as a notation to emphasise this in future text $U \gg v, U \gg w$.

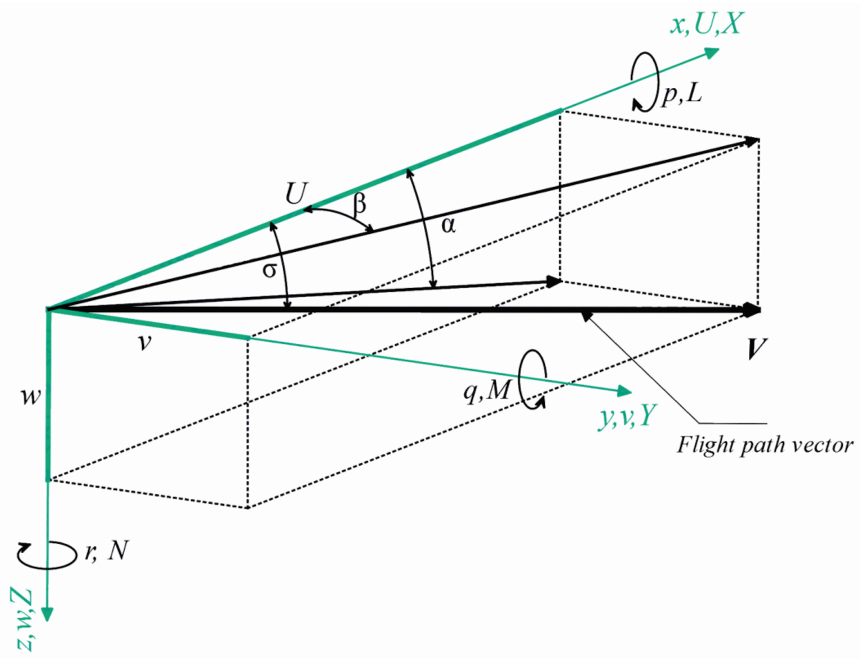


FIGURE 4-7 MISSILE AERODYNAMIC FORCES, MOMENTS, COORDINATES, AND VELOCITY COMPONENTS IN BODY-FIXED SYSTEM

If we project the velocity vector onto the x, z -plane, the angle of attack is obtained as the angle between this projection and x -axis. Similarly, we use the definition of sideslip angle β as the angle between the projection of velocity vector onto x, y -plane and x - axis. The angle σ , known as the total angle of attack, is measured in the plane common to the x -axis and the velocity vector. From the geometry of Figure 4-7, we have

$$\tan\alpha = \frac{w}{U}, \tan\beta = \frac{v}{U}, \cos\sigma = \frac{U}{V}, V^2 = U^2 + v^2 + w^2 \tag{4.13}$$

and

$$\sigma = \tan^{-1}\left[(\tan\alpha)^2 + (\tan\beta)^2\right] \tag{4.14}$$

This is a tangent definition of the angle of attack, and it is commonly used in the aerodynamics and flight dynamics of missiles.

If α , β and σ are sufficiently small, this expression may be approximated by

$$\sigma = \sqrt{\alpha^2 + \beta^2} \tag{4.15}$$

where to the same order of accuracy

$$\alpha = \frac{w}{U}, \beta = \frac{v}{U}, U = V \tag{4.16}$$

The components of the resultant aerodynamic force and moment in the body-fixed reference frame (Figure 4-7) can be expressed in terms of dimensionless coefficients, dynamic pressure, reference area, and characteristic length as follows:

Axial force: $X = C_X QS$ (4.17)

Side force: $Y = C_Y QS$ (4.18)

Normal force: $Z = C_Z qS$ (4.19)

Rolling moment: $L = C_l QSl$ (4.20)

Pitching moment: $M = C_m QSl$ (4.21)

Yawing moment: $N = C_n QSl$ (4.22)

Because of the orientation of body axes, we usually present aerodynamic characteristics of a missile by using axial (C_A) and normal (C_N) force coefficients:

$$C_A = -C_X > 0, C_N = -C_Z > 0 \tag{4.23}$$

In general, the standard six-degree-of-freedom aerodynamic coefficients $C_X, C_Y, C_Z, C_l, C_m, C_n$ are primarily a function of centre-of-gravity location, altitude (air density), Mach number (M), Reynolds number (R_e), angle of attack (α), and sideslip angle (β), and are secondary functions of the time rate of change of angle of attack ($\dot{\alpha}$) and sideslip ($\dot{\beta}$), and the angular velocity of the missile (p, q, r).

The Mach number is a non-dimensional number defined as the ratio of missile airspeed to speed of sound $\left(M = \frac{V}{a} \right)$.

The Reynolds number is a non-dimensional number defined as $R_e = \frac{\rho V l}{\mu} = \frac{Vl}{\nu}$.

where

ρ the density of the fluid,

μ the coefficient of absolute viscosity of the fluid,

V the velocity,

l the characteristic length, and

$\nu = \frac{\mu}{\rho}$ is the kinematical viscosity.