Машински факултет
УНИВЕРЗИТЕТА У БЕОГРАДУ

Динамика роботског система- Ланграњжеве једначине друге врсте у коваријантном облику

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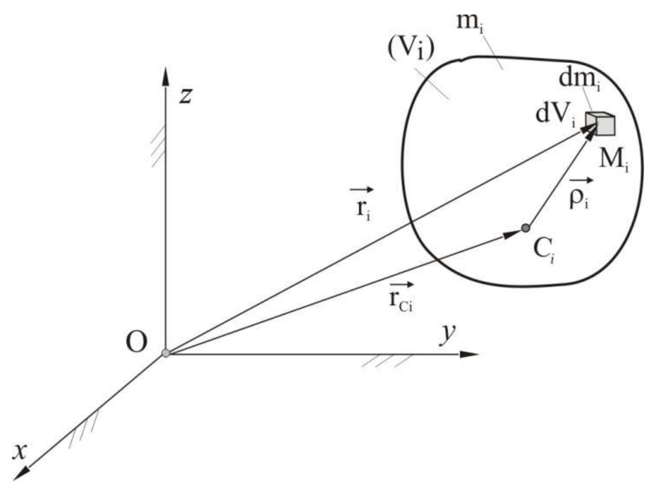
Кинетичка енергија [Vi] роботског сегмента и РС

$$dE_{k(i)} = \frac{1}{2} dm_i \, v_{M_i}^2$$



$$E_{k(i)} = \frac{1}{2} m_i v_{C_i}^2 + \frac{1}{2} \left(\vec{\omega}_i \right) \left[J_{C_i} \right] \left\{ \vec{\omega}_i \right\}$$

Екtr Кинетичка енергија транслације

$$E_{k(i)tr} = \frac{1}{2} m_i v_{C_i}^2$$

$$\vec{v}_{C_i} = \sum_{\alpha=1}^{n} \vec{T}_{\alpha(i)} \dot{q}^{\alpha}$$

$$E_k^{(tr)} = \sum_{i=1}^{n} E_k^{(i,tr)} = \frac{1}{2} \sum_{i=1}^{n} m_i v_i^2 = \frac{1}{2} \sum_{i=1}^{n} m_i \vec{v}_i \cdot \vec{v}_i$$

$$\vec{v}_i \, (= \vec{v}_{C_i}) = \sum_{\alpha=1}^{n} \frac{\partial \vec{r}_i}{\partial q^{\alpha}} \dot{q}^{\alpha} \rightarrow E_k^{(tr)} = \frac{1}{2} \sum_{i=1}^{n} m_i \left(\sum_{\alpha=1}^{n} \frac{\partial \vec{r}_i}{\partial q^{\alpha}} \dot{q}^{\alpha} \right) \cdot \left(\sum_{\beta=1}^{n} \frac{\partial \vec{r}_i}{\partial q^{\beta}} \dot{q}^{\beta} \right)$$

$$a_{\alpha\beta}^{(tr)} = \sum_{i=1}^{n} m_i \frac{\partial \vec{r}_i}{\partial q^\alpha} \cdot \frac{\partial \vec{r}_i}{\partial q^\beta} \rightarrow a_{\alpha\beta}^{(tr)} = a_{\beta\alpha}^{(tr)} \quad ; \quad \vec{T}_{i(\alpha)} = \frac{\partial \vec{r}_i}{\partial q^\alpha} \Rightarrow$$

$$a_{\alpha\beta}^{(tr)} = \sum_{i=1}^{n} m_i \, \vec{T}_{i(\alpha)} \cdot \vec{T}_{i(\beta)}$$

$$E_{ki} = \frac{1}{2} m_i \left(\sum_{\alpha=1}^{n} \vec{T}_{\alpha(i)} \dot{q}^\alpha \right) \cdot \left\{ \sum_{\beta=1}^{n} \vec{T}_{\beta(i)} \dot{q}^\beta \right\} \Rightarrow E_{ki}tr = \frac{1}{2} \sum_{\alpha=1}^{n} \sum_{\beta=1}^{n} a^{tr}{}_{\alpha\beta(i)} \dot{q}^\alpha \dot{q}^\beta$$

$$a^{tr}{}_{\alpha\beta(i)} = m_i \left(\vec{T}_{\alpha(i)} \right) \left\{ \vec{T}_{\beta(i)} \right\}$$

Ек рот Кинетичка енергија ротације

$$E_{k(i)rot} = \frac{1}{2} \left(\vec{\omega}_i \right) \left[J_{C_i} \right] \left\{ \vec{\omega}_i \right\} \Leftarrow \vec{\omega}_i = \sum_{\alpha=1}^{n} \vec{\Omega}_{\alpha(i)} \dot{q}^\alpha$$

$$E_{ki}rot = \frac{1}{2} \left(\sum_{\alpha=1}^{n} \vec{\Omega}_{\alpha(i)} \dot{q}^\alpha \right) \left[J_{C_i} \right] \left\{ \sum_{\beta=1}^{n} \vec{\Omega}_{\beta(i)} \dot{q}^\beta \right\},$$

$$E_{ki}rot = \frac{1}{2} \sum_{\alpha=1}^{n} \sum_{\beta=1}^{n} a^{rot}{}_{\alpha\beta(i)} \dot{q}^\alpha \dot{q}^\beta$$

$$a^{rot}{}_{\alpha\beta(i)} = \left(\vec{\Omega}_{\alpha(i)} \right) \left[J_{C_i} \right] \left\{ \vec{\Omega}_{\beta(i)} \right\}$$

$$E_{ki} = \frac{1}{2} \sum_{\alpha=1}^{n} \sum_{\beta=1}^{n} a_{\alpha\beta(i)} \dot{q}^{\alpha} \dot{q}^{\beta}$$

где су

$$a_{\alpha\beta_{(i)}} = a^{tr}{}_{\alpha\beta_{\,(i)}} + a^{rot}{}_{\alpha\beta_{\,(i)}} = m_i \left(\vec{T}_{\alpha(i)} \right) \left\{ \vec{T}_{\beta(i)} \right\} + \left(\vec{\Omega}_{\alpha(i)} \right) \left[J_{Ci} \right] \left\{ \vec{\Omega}_{\beta(i)} \right\}$$

$$a_{\alpha\beta(i)} = a_{\beta\alpha(i)}$$

Eк роботског система

$$E_k = \sum_{i=1}^{n} E_{k(i)}$$

$$E_k = \frac{1}{2} \sum_{\alpha=1}^{n} \sum_{\beta=1}^{n} \sum_{i=1}^{n} a_{\alpha\beta(i)} \dot{q}^{\alpha} \dot{q}^{\beta}$$

$\left(a_{\alpha\beta} = a_{\alpha\beta}^{(tr)} + a_{\alpha\beta}^{(rot)} \right)$

$$a_{\alpha\beta} = \sum_{i=1}^{n} a_{\alpha\beta(i)} = \sum_{i=1}^{n} m_i \left(\vec{T}_{\alpha(i)} \right) \left\{ \vec{T}_{\beta(i)} \right\} + \sum_{i=1}^{n} \left(\vec{\Omega}_{\alpha(i)} \right) \left[J_{Ci} \right] \left\{ \vec{\Omega}_{\beta(i)} \right\}$$

$$a_{\alpha\beta} = a_{\beta\alpha}$$

,

$$E_k = \frac{1}{2} \sum_{\alpha=1}^{n} \sum_{\beta=1}^{n} a_{\alpha\beta} \dot{q}^{\alpha} \dot{q}^{\beta}$$

Коваријантне координате основног метричког тензора

- Основни метрички тензор $[a_{\alpha\beta}] \in R^{n \times n}$

- позитивно дефинитна квадратна форма генералисаних брзина

$$E_k(\sum_{i=1}^{n} (\dot{q}^i)^2 \neq 0) > 0, E_k(\sum_{i=1}^{n} (\dot{q}^i)^2 = 0) = 0,$$

$$E_k = \frac{1}{2}(\dot{q})\left[a_{\alpha\beta}\right]\{\dot{q}\}$$

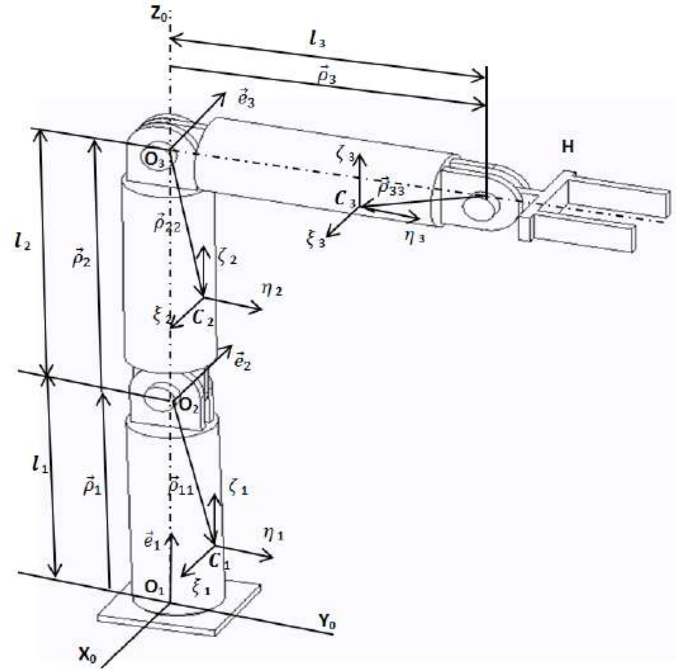
- .

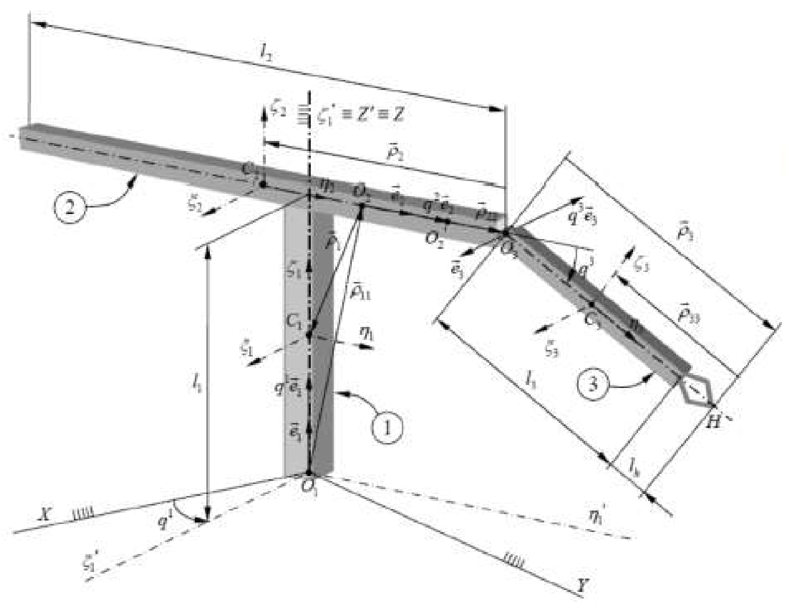
Пример РС- 3сс

$$\mathrm{A}(\boldsymbol{q}) = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$a_{\alpha\beta} = \sum_{i=\sup(\alpha,\beta)}^{n} m_i(\vec{T}_{\alpha(i)})\{\vec{T}_{\beta(i)}\} + \sum_{i=\sup(\alpha,\beta)}^{n} \bar{\xi}_\alpha \bar{\xi}_\beta (\vec{e}_{\alpha(i)})[J_{ci}]\{\vec{e}_{\beta(i)}\}$$

$$a_{13} = m_3(\vec{T}_{1(3)})\{\vec{T}_{3(3)}\} + \bar{\xi}_1\bar{\xi}_3([\mathrm{A}_{13}]^{\mathrm{T}}\{\vec{e}_{1(3)}\})[J_{c3}]\{\vec{e}_{3(3)}\}$$





$$a_{11} = \sum_{i=1}^{3} a_{11,i}^{tr} + a_{11,i}^{rot} = m_2\left(\frac{l_2}{2} - d_1 - q^2\right)^2 + m_3\left(d_1 + q^2 + \frac{l_3}{2}\cos q^3\right)^2 +$$

$$+ J_{\zeta_1} + J_{\zeta_2} + \frac{1}{2}\left(J_{\eta_3} + J_{\zeta_3}\right) + \frac{1}{2}\left(J_{\zeta_3} - J_{\eta_3}\right)\cos 2q^3$$

$$a_{12} = \sum_{i=1}^{3} a_{12,i}^{tr} + a_{12,i}^{rot} = 0,\ a_{22} = \sum_{i=1}^{3} a_{22,i}^{tr} + a_{22,i}^{rot} = m_2 + m_3,\ a_{13} = \sum_{i=1}^{3} a_{13,i}^{tr} + a_{13,i}^{rot} = 0$$

$$a_{23} = \sum_{i=1}^{3} a_{23,i}^{tr} + a_{23,i}^{rot} = -\frac{m_3 l_3}{2}\sin q^3,\ a_{33} = \sum_{i=1}^{3} a_{33,i}^{tr} + a_{33,i}^{rot} = \frac{1}{4} m_3 l_3^2 + J_{\xi_3}$$

$$A(\mathbf{q}) = \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & a_{23} \\ 0 & a_{23} & a_{33} \end{bmatrix}$$

```
Ek1= 0.5*m1*(Vc11'*Vc11)+ 0.5*w11'*J1*w11;
Ek2= 0.5*m2*(Vc21'*Vc21)+ 0.5*w22'*J2*w22;
Ek3= 0.5*m3*(Vc31'*Vc31)+ 0.5*w33'*J3*w33;
Ek=Ek1 + Ek2 + Ek3;
```

$$E_k = \frac{1}{2}\sum_{\alpha=1}^{n}\sum_{\beta=1}^{n} a_{\alpha\beta}\dot{q}^{\alpha}\dot{q}^{\beta}$$

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%Koeficijenti metrickog tenzora
a11=diff(diff(Ek,q1d),q1d);
a12=diff(diff(Ek,q1d),q2d);
a13=diff(diff(Ek,q1d),q3d);
a22=diff(diff(Ek,q2d),q2d);
a23=diff(diff(Ek,q2d),q3d);
a33=diff(diff(Ek,q3d),q3d);
a21=a12; a31=a13; a32=a23;
A=[a11 a12 a13; a21 a22 a23; a31 a32 a33];
```

$$a_{\alpha\beta} = \frac{\partial^2 E_k}{\partial\dot{q}^{\alpha}\partial\dot{q}^{\beta}}, \quad \Rightarrow a_{\alpha\beta} = a_{\beta\alpha}.$$

Коваријантни облик диференцијалних кретања РС

- Лагранжеве једначине друге врсте:

$(q^1, q^2, \dots, q^n)$

$$\frac{d}{dt}\left(\frac{\partial E_k}{\partial \dot{q}^\gamma}\right) - \frac{\partial E_k}{\partial q^\gamma} = Q_\gamma, \quad \gamma = 1,2,\dots,n$$

$$\frac{\partial E_k}{\partial \dot{q}^\gamma} = \frac{1}{2}\sum_{\beta=1}^{n} a_{\gamma\beta}\dot{q}^\beta + \frac{1}{2}\sum_{\alpha=1}^{n} a_{\alpha\gamma}\dot{q}^\alpha \qquad \frac{\partial E_k}{\partial \dot{q}^\gamma} = \sum_{\alpha=1}^{n} a_{\alpha\gamma}\dot{q}^\alpha$$

$$\sum_{\alpha=1}^{n}\sum_{\beta=1}^{n}\frac{\partial a_{\alpha\gamma}}{\partial q^\beta}\dot{q}^\alpha\dot{q}^\beta = \frac{1}{2}\sum_{\alpha=1}^{n}\sum_{\beta=1}^{n}\frac{\partial a_{\alpha\gamma}}{\partial q^\beta}\dot{q}^\alpha\dot{q}^\beta + \frac{1}{2}\sum_{\alpha=1}^{n}\sum_{\beta=1}^{n}\frac{\partial a_{\beta\gamma}}{\partial q^\alpha}\dot{q}^\alpha\dot{q}^\beta)$$

$$\frac{d}{dt}\left(\frac{\partial E_k}{\partial \dot{q}^\gamma}\right) = \sum_{\alpha=1}^{n} a_{\alpha\gamma}\ddot{q}^\alpha + \sum_{\alpha=1}^{n}\sum_{\beta=1}^{n}\frac{\partial a_{\alpha\gamma}}{\partial q^\beta}\dot{q}^\alpha\dot{q}^\beta$$

$$\frac{d}{dt}\left(\frac{\partial E_k}{\partial \dot{q}^\gamma}\right) = \sum_{\alpha=1}^{n} a_{\alpha\gamma}\ddot{q}^\alpha + \frac{1}{2}\sum_{\alpha=1}^{n}\sum_{\beta=1}^{n}\left(\frac{\partial a_{\alpha\gamma}}{\partial q^\beta} + \frac{\partial a_{\beta\gamma}}{\partial q^\alpha}\right)\dot{q}^\alpha\dot{q}^\beta$$

- $$\frac{\partial E_k}{\partial q^\gamma} = \frac{1}{2}\sum_{\alpha=1}^{n}\sum_{\beta=1}^{n}\frac{\partial a_{\alpha\gamma}}{\partial q^\beta}\dot{q}^\alpha\dot{q}^\beta$$

$$\sum_{\alpha=1}^{n} a_{\alpha\gamma}\ddot{q}^\alpha + \sum_{\alpha=1}^{n}\sum_{\beta=1}^{n}\Gamma_{\alpha\beta,\gamma}\dot{q}^\alpha\dot{q}^\beta = Q_\gamma$$

Коваријантни облик,
Кристофелов симбол 1 врсте

$$\Gamma_{\alpha\beta,\gamma} = \frac{1}{2}\left(\frac{\partial a_{\beta\gamma}}{\partial q^\alpha} + \frac{\partial a_{\gamma\alpha}}{\partial q^\beta} - \frac{\partial a_{\alpha\beta}}{\partial q^\gamma}\right)$$

$$\Gamma_{\alpha\beta,\gamma} = \Gamma_{\beta\alpha,\gamma}, \quad \alpha,\beta,\gamma = 1,2,\dots,n$$

- Лагранжеве једначине друге врсте - **коваријантни облик :**

$$\sum_{\alpha=1}^{n} a_{\alpha\gamma}\ddot{q}^{\alpha} + \sum_{\alpha=1}^{n}\sum_{\beta=1}^{n}\Gamma_{\alpha\beta,\gamma}\dot{q}^{\alpha}\dot{q}^{\beta} = Q_{\gamma}$$

- $a_{\gamma\alpha}$ - „коефицијенти“ метричког тензора;
- $\Gamma_{\alpha\beta,\gamma}$ Кристофелови симболи прве врсте;
- Q_{γ}- генералисане силе система активних сила које делују на роботски систем

Матрична форма Лан. јед

$$\mathrm{K}_{\alpha\beta,\gamma} = \frac{1}{2}\left(\frac{\partial a_{\beta\gamma}}{\partial q_{\alpha}} + \frac{\partial a_{\alpha\gamma}}{\partial q_{\beta}} - \frac{\partial a_{\alpha\beta}}{\partial q_{\gamma}}\right)$$

$$A(\mathbf{q})\ddot{\mathbf{q}} + \mathrm{K}(\mathbf{q},\dot{\mathbf{q}}) = \mathbf{Q}$$

$$A(\mathbf{q}) = \left[a_{\alpha\beta}\right]_{n\times n}$$

$$K(\mathbf{q},\dot{\mathbf{q}}) = \left[\dot{\mathbf{q}}^{\mathrm{T}}\mathrm{K}_{1}\dot{\mathbf{q}},\ldots,\dot{\mathbf{q}}^{\mathrm{T}}\mathrm{K}_{n}\dot{\mathbf{q}}\right]$$

$K_r \rightarrow \gamma$

Кристофелови симболи 1 врсте

$$\Gamma_{\alpha\beta,\gamma}=\frac{1}{2}\left(\frac{\partial a_{\beta\gamma}}{\partial q^{\alpha}}+\frac{\partial a_{\gamma_{\alpha}}}{\partial q^{\beta}}-\frac{\partial a_{\alpha\beta}}{\partial q^{\gamma}}\right)$$

$$\Gamma_{\alpha\beta,\gamma}=\Gamma_{\beta\alpha,\gamma},\quad \alpha,\beta,\gamma=1,2,\ldots,n$$

$$\Gamma_{\alpha\beta,\gamma}=-\Gamma_{\alpha\gamma,\beta},\quad \alpha,\beta,\gamma=1,2,\ldots,n$$

Симетричан по прва 2 индекса, и антисиметричан по друга 2 индекса

Аналитички израз за Кристофелов симбол 1 врсте

$$a_{\alpha\beta}=\sum_{i=1}^{n}m_i\frac{\partial\vec{r}_{C_i}}{\partial q^{\alpha}}\cdot\frac{\partial\vec{r}_{C_i}}{\partial q^{\beta}}+\sum_{i=1}^{n}\int_{(Vi)}\frac{\partial\vec{\rho}_i}{\partial q^{\alpha}}\cdot\frac{\partial\vec{\rho}_i}{\partial q^{\beta}}dm_i \qquad \frac{\partial\vec{r}_{C_i}}{\partial q^{\alpha}}=\vec{T}_{\alpha(i)},\ \frac{\partial\vec{\rho}_i}{\partial q^{\alpha}}=\vec{\Omega}_{\alpha(i)}\times\vec{\rho}_i,\ \left[J_{C_i}\right]=-\int_{(Vi)}\left[\rho_i^d\right]^2dm_i$$

$$\frac{\partial a_{\beta\gamma}}{\partial q^{\alpha}}=\sum_{i=1}^{n}m_i\left(\frac{\partial^2\vec{r}_{C_i}}{\partial q^{\alpha}\partial q^{\beta}}\cdot\frac{\partial\vec{r}_{C_i}}{\partial q^{\gamma}}+\frac{\partial\vec{r}_{C_i}}{\partial q^{\beta}}\cdot\frac{\partial^2\vec{r}_{C_i}}{\partial q^{\alpha}\partial q^{\gamma}}\right)+\sum_{i=1}^{n}\int_{(Vi)}\left(\frac{\partial^2\vec{\rho}_i}{\partial q^{\alpha}\partial q^{\beta}}\cdot\frac{\partial\vec{\rho}_i}{\partial q^{\gamma}}+\frac{\partial\vec{\rho}_i}{\partial q^{\beta}}\cdot\frac{\partial^2\vec{\rho}_i}{\partial q^{\alpha}\partial q^{\gamma}}\right)dm_i$$

$$\Gamma_{\alpha\beta,\gamma}=\sum_{i=1}^{n}m_i\frac{\partial^2\vec{r}_{C_i}}{\partial q^{\alpha}\partial q^{\beta}}\cdot\frac{\partial\vec{r}_{C_i}}{\partial q^{\gamma}}+\sum_{i=1}^{n}\int_{(Vi)}\frac{\partial^2\vec{\rho}_i}{\partial q^{\alpha}\partial q^{\beta}}\cdot\frac{\partial\vec{\rho}_i}{\partial q^{\gamma}}dm_i$$

$$\frac{\partial\vec{r}_{C_i}}{\partial q^{\alpha}}=\vec{T}_{\alpha(i)},\ \frac{\partial\vec{\rho}_i}{\partial q^{\alpha}}=\vec{\Omega}_{\alpha(i)}\times\vec{\rho}_i,\ \left[J_{C_i}\right]=-\int_{(Vi)}\left[\rho_i^d\right]^2dm_i$$

$$a_{\alpha\beta}=\sum_{i=1}^{n}m_i\frac{\partial\vec{r}_{C_i}}{\partial q^{\alpha}}\cdot\frac{\partial\vec{r}_{C_i}}{\partial q^{\beta}}+\sum_{i=1}^{n}\int_{(Vi)}\frac{\partial\vec{\rho}_i}{\partial q^{\alpha}}\cdot\frac{\partial\vec{\rho}_i}{\partial q^{\beta}}dm_i$$

$$\Gamma_{\alpha\beta,\gamma} = \Gamma^{tr}{}_{\alpha\beta,\gamma} + \Gamma^{rot}{}_{\alpha\beta,\gamma}$$

$$\Gamma^{tr}{}_{\alpha\beta,\gamma} = \sum_{i=1}^{n} m_i \frac{\partial^2 \vec{r}_{C_i}}{\partial q^\alpha \partial q^\beta} \frac{\partial \vec{r}_{C_i}}{\partial q^\gamma}, \quad \alpha \le \beta$$

$$\vec{T}_{\alpha(i)} = \frac{\partial \vec{r}_i}{\partial q^\alpha}, \quad \vec{T}_{\beta(i)} = \frac{\partial \vec{r}_i}{\partial q^\beta},$$

$$\vec{T}_{\alpha(i)} = \frac{\partial \vec{r}_i}{\partial q^\alpha} = \bar{\xi}_\alpha \vec{e}_\alpha \times \vec{R}_{\alpha(i)} + \xi_\alpha \vec{e}_\alpha$$

$$\vec{R}_{\alpha(i)} = \sum_{k=\alpha}^{i} \left(\vec{\rho}_{kk} + \xi_k \vec{e}_k q^k \right) + \vec{\rho}_i$$

$$\frac{\partial^2 \vec{r}_{C_i}}{\partial q^\alpha \partial q^\beta} = \frac{\partial \vec{T}_{\alpha(i)}}{\partial q^\beta}, \quad \alpha \le \beta$$

$$\frac{\partial \vec{r}_{ci}}{\partial q^\gamma} = \vec{T}_{\gamma(i)}$$

$$\frac{\partial \vec{T}_{\alpha(i)}}{\partial q^\beta} = \frac{\partial}{\partial q^\beta} \left(\bar{\xi}_\alpha \vec{e}_\alpha \times \vec{R}_{\alpha(i)} + \xi_\alpha \vec{e}_\alpha \right), \quad \alpha \le \beta$$

$$= \frac{\partial}{\partial q^\beta} \left(\bar{\xi}_\alpha \vec{e}_\alpha \right) \times \vec{R}_{\alpha(i)} + \left(\bar{\xi}_\alpha \vec{e}_\alpha \right) \times \frac{\partial \vec{R}_{\alpha(i)}}{\partial q^\beta} + \frac{\partial \left(\xi_\alpha \vec{e}_\alpha \right)}{\partial q^\beta}$$

0 0

$$\frac{\partial \vec{T}_{\alpha(i)}}{\partial q^\beta} = \left(\bar{\xi}_\alpha \vec{e}_\alpha \right) \times \frac{\partial \vec{R}_{\alpha(i)}}{\partial q^\beta} =$$

$$= \bar{\xi}_\alpha \vec{e}_\alpha \times \frac{\partial}{\partial q^\beta} \left(\sum_{k=1}^{\alpha-1} \left(\vec{\rho}_{kk} + \xi_k \vec{e}_k q^k \right) + \vec{\rho}_{\alpha-1} + \sum_{k=\alpha}^{i} \left(\vec{\rho}_{kk} + \xi_k \vec{e}_k q^k \right) + \vec{\rho}_i \right)$$

0

$$\frac{\partial \vec{T}_{\alpha(i)}}{\partial q^\beta} = \left(\bar{\xi}_\alpha \vec{e}_\alpha \right) \times \frac{\partial \vec{r}_{(i)}}{\partial q^\beta}$$

$$\frac{\partial \vec{T}_{\alpha(i)}}{\partial q^{\beta}} = \left(\bar{\xi}_{\alpha}\vec{e}_{\alpha}\right) \times \vec{T}_{\beta(i)}$$

$$\Gamma^{tr}{}_{\alpha\beta,\gamma} = \sum_{i=1}^{n} m_i \left(\bar{\xi}_{\alpha}\vec{e}_{\alpha} \times \vec{T}_{\beta(i)}\right) \cdot \vec{T}_{\gamma(i)}, \quad \alpha \le \beta$$

2x

$$\Gamma^{tr}{}_{\alpha\beta,\gamma} = \sum_{i=1}^{n} m_i \bar{\xi}_{\alpha}\vec{e}_{\alpha} \cdot \left(\vec{T}_{\beta(i)} \times \vec{T}_{\gamma(i)}\right), \quad \alpha \le \beta$$

$T_{\beta(i)} = 0$, $T_{\gamma(i)}$, $i > \beta, \gamma, i$

$$\Gamma^{tr}{}_{\alpha\beta,\gamma} = \sum_{i=\sup(\beta,\gamma)}^{n} m_i \bar{\xi}_{\alpha}\vec{e}_{\alpha} \cdot \left(\vec{T}_{\beta(i)} \times \vec{T}_{\gamma(i)}\right), \quad \alpha \le \beta$$

$$\Gamma^{rot}{}_{\alpha\beta,\gamma} = \sum_{i=1}^{n} \int_{(Vi)} \frac{\partial^2 \vec{\rho}_i}{\partial q^{\alpha} \partial q^{\beta}} \frac{\partial \vec{\rho}_i}{\partial q^{\gamma}} dm_i \quad \alpha \le \beta$$

$$\frac{\partial \vec{\rho}_i}{\partial q^{\gamma}} = \vec{\Omega}_{\gamma(i)} \times \vec{\rho}_i$$

$$\frac{\partial^2 \vec{\rho}_i}{\partial q^{\alpha} \partial q^{\beta}} = \frac{\partial}{\partial q^{\beta}}\left(\frac{\partial \vec{\rho}_i}{\partial q^{\alpha}}\right) = \frac{\partial}{\partial q^{\beta}}\left(\vec{\Omega}_{\alpha(i)} \times \vec{\rho}_i\right) = \frac{\partial}{\partial q^{\beta}}\left(\vec{\Omega}_{\alpha(i)}\right) \times \vec{\rho}_i + \vec{\Omega}_{\alpha(i)} \times \frac{\partial \vec{\rho}_i}{\partial q^{\beta}}$$

0

$$= \vec{\Omega}_{\alpha(i)} \times \frac{\partial \vec{\rho}_i}{\partial q^{\beta}} = \vec{\Omega}_{\alpha(i)} \times \left(\vec{\Omega}_{\beta(i)} \times \vec{\rho}_i\right)$$

$$\left(\vec{\Omega}_{\alpha(i)} \times \left(\vec{\Omega}_{\beta(i)} \times \vec{\rho}_i\right)\right) \cdot \left(\vec{\Omega}_{\gamma(i)} \times \vec{\rho}_i\right)$$

A X B X C

$$\vec{A} \times \left(\vec{B} \times \vec{C}\right) = \begin{vmatrix} \vec{B} & \vec{C} \\ \left(\vec{A} \cdot \vec{B}\right) & \left(\vec{A} \cdot \vec{C}\right) \end{vmatrix} = \vec{B}\left(\vec{A} \cdot \vec{C}\right) - \vec{C}\left(\vec{A} \cdot \vec{B}\right)$$

$$\vec{A}\times\left(\vec{B}\times\vec{C}\right)=\begin{vmatrix}\vec{B} & \vec{C}\\ \left(\vec{A}\cdot\vec{B}\right) & \left(\vec{A}\cdot\vec{C}\right)\end{vmatrix}=\vec{B}\left(\vec{A}\cdot\vec{C}\right)-\vec{C}\left(\vec{A}\cdot\vec{B}\right)$$

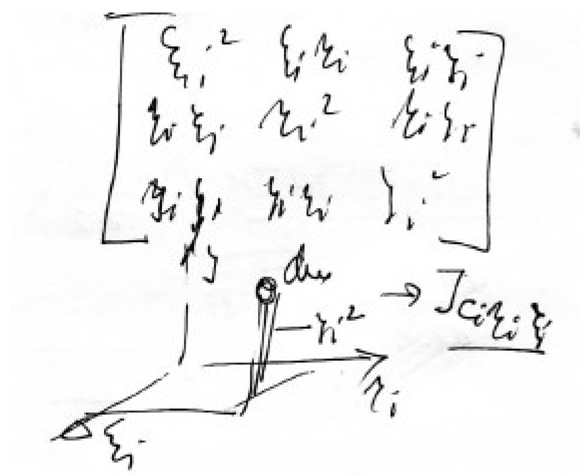
$$\left(\vec{\Omega}_{\beta(i)}\left(\vec{\Omega}_{\alpha(i)}\cdot\vec{\rho}_i\right)-\vec{\rho}_i\left(\vec{\Omega}_{\alpha(i)}\cdot\vec{\Omega}_{\beta(i)}\right)\right)\cdot\left(\vec{\Omega}_{\gamma(i)}\times\vec{\rho}_i\right)$$

$$\left(\vec{\Omega}_{\beta(i)}\left(\vec{\Omega}_{\alpha(i)}\cdot\vec{\rho}_i\right)\right)\cdot\left(\vec{\Omega}_{\gamma(i)}\times\vec{\rho}_i\right)$$

$$\left(\vec{\Omega}_{\alpha(i)}\cdot\vec{\rho}_i\right)\vec{\rho}_i\cdot\left(\vec{\Omega}_{\beta(i)}\times\vec{\Omega}_{\gamma(i)}\right)$$

$$\left(\vec{\Omega}_{\beta(i)}\times\vec{\Omega}_{\gamma(i)}\right)\left\{\vec{\rho}_i\right\}\left(\vec{\rho}_i\right)\left\{\vec{\Omega}_{\alpha(i)}\right\}$$

$$\left\{\vec{\rho}_i\right\}\left(\vec{\rho}_i\right)=\begin{Bmatrix}\xi_i\\ \eta_i\\ \zeta_i\end{Bmatrix}\begin{pmatrix}\xi_i & \eta_i & \zeta_i\end{pmatrix}=\begin{bmatrix}\xi_i^2 & \xi_i\eta_i & \xi_i\zeta_i\\ \eta_i\xi_i & \eta_i^2 & \eta_i\zeta_i\\ \zeta_i\xi_i & \zeta_i\eta_i & \zeta_i^2\end{bmatrix}$$



$$\left[\Pi_{Ci}\right]=\int_V\left\{\vec{\rho}_i\right\}\left(\vec{\rho}_i\right)dm_i$$

Планарни тензор инерције

$$[\Pi_{Ci}] = \begin{bmatrix} \Pi_{Ci\eta_i\zeta_i} & J_{\xi_i\eta_i} & J_{\xi_i\zeta_i} \\ J_{\xi_i\eta_i} & \Pi_{Ci\xi_i\zeta_i} & J_{\eta_i\zeta_i} \\ J_{\xi_i\zeta_i} & J_{\eta_i\zeta_i} & \Pi_{Ci\xi_i\eta_i} \end{bmatrix}$$

$$\Gamma^{rot}{}_{\alpha\beta,\gamma} = \sum_{i=1}^{n} \left(\vec{\Omega}_{\beta(i)} \times \vec{\Omega}_{\gamma(i)}\right)[\Pi_{Ci}]\left\{\vec{\Omega}_{\alpha(i)}\right\}$$

$$\Gamma^{rot}{}_{\alpha\beta,\gamma} = \sum_{i=\sup(\beta,\gamma)}^{n} \left(\vec{\Omega}_{\beta(i)} \times \vec{\Omega}_{\gamma(i)}\right)[\Pi_{Ci}]\left\{\vec{\Omega}_{\alpha(i)}\right\}$$

$$\Gamma^{rot}{}_{\alpha\beta,\gamma} = \sum_{i=\sup(\beta,\gamma)}^{n} \bar{\xi}_{\alpha}\bar{\xi}_{\beta}\bar{\xi}_{\gamma}\left(\vec{e}_{\beta} \times \vec{e}_{\gamma}\right)[\Pi_{Ci}]\left\{\vec{e}_{\alpha}\right\}$$

$$\left(\vec{e}_{\beta} \times \vec{e}_{\gamma}\right) = -\left(\vec{e}_{\gamma} \times \vec{e}_{\beta}\right)$$

$$\Gamma^{rot}{}_{\alpha\beta,\gamma} = -\Gamma^{rot}{}_{\alpha\gamma,\beta}$$

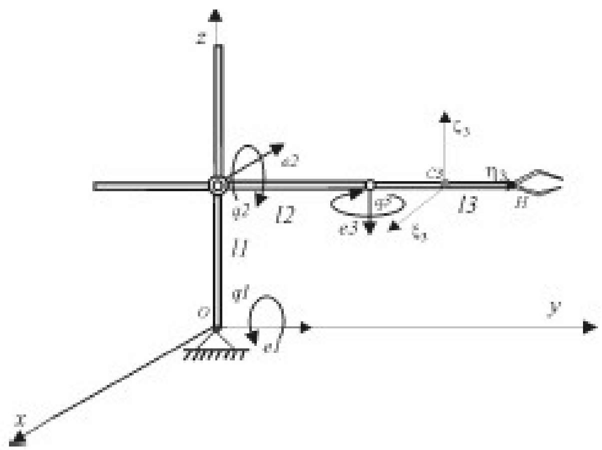
Пример израчунавања планарног тензора инерције

Primer 3. *Odrediti planarni tenzor inercije* Π_C *segmenta* $[V_3]$ *datog robotskog sistema (primer1), koji je oblika prizmatičnog štapa, mase* $m_3 = 3\ kg$ *i dužine* $l_3 = 2\ m$.

$$J_{\eta 3 C_3 \zeta 3} = \frac{(J_{C_3\eta} + J_{C_3\zeta} - J_{C_3\xi})}{2}, \quad J_{\zeta_3 C_3 \xi_3} = \frac{(J_{C_3\xi} + J_{C_3\zeta} - J_{C_3\eta})}{2},$$

$$J_{\xi_3 C_3 \eta_3} = \frac{(J_{C_3\xi} + J_{C_3\eta} - J_{C_3\zeta})}{2}$$

$$J_{C_3\xi} = J_{C_3\zeta} = \frac{m_3 l_3^2}{12} = \frac{3 \cdot 2^2}{12} = 1 \quad kgm^2, \qquad J_{C_3\eta} = 0 \quad kgm^2$$



$$J_{\eta 3 C_3 \zeta 3} = \frac{(J_{C_3\eta} + J_{C_3\zeta} - J_{C_3\xi})}{2} = 0 \quad kgm^2,$$

$$J_{\zeta_3 C_3 \xi_3} = \frac{(J_{C_3\xi} + J_{C_3\zeta} - J_{C_3\eta})}{2} = 1 \quad kgm^2,$$

$$J_{\xi_3 C_3 \eta_3} = \frac{(J_{C_3\xi} + J_{C_3\eta} - J_{C_3\zeta})}{2} = 0 \quad kgm^2,$$

$$\left[\Pi_{C3}^{(3)}\right] = \begin{bmatrix} J_{\eta 3C3\zeta 3} & 0 & 0 \\ 0 & J_{\zeta 3C3\xi 3} & 0 \\ 0 & 0 & J_{\xi 3C3\eta 3} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$