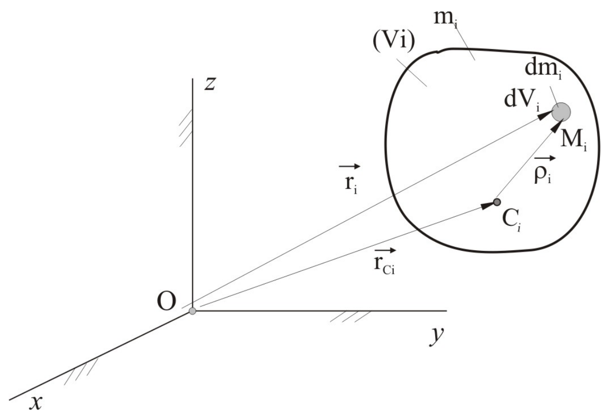
7.3 Kinetička energija robotskog sistema

Razmatramo robotski sistem sa n segmenata $(V_1),(V_2),...,(V_n)$ koji oblik otvorenog kinematičkog lanca bez grananja. Diferencijal kinetičke energije robotskog segmenta (V_i), mase m_i, iznosi (vidi sl. 7.6)



Slika 7.6

$$dE_{ki} = \frac{1}{2} dm v_{Mi}^2 , \tag{7.92}$$

gde je brzina unutrašnje tačke M_i elementarne zapremine dV_i, kojoj odgovara elementarna masa dm_i, jednaka (C_i-centar inercije segmenta(V_i), $\vec{\omega}_i$-ugaona brzina segmenta (V_i)):

$$\vec{v}_{Mi} = \vec{v}_i = \vec{v}_{Ci} + \vec{\omega}_i \times \vec{\rho}_i . \tag{7.93}$$

Kinetička energija segmenta iznosi

$$E_{ki} = \frac{1}{2} \int_{(V_i)} (\vec{v}_{Ci} + \vec{\omega}_i \times \vec{\rho}_i) \cdot (\vec{v}_{Ci} + \vec{\omega}_i \times \vec{\rho}_i) dm_i . \tag{7.94}$$

Poslednji izraz može da se dovede na oblik

$$E_{ki} = \frac{1}{2} \vec{v}_{Ci}^2 \int_{(Vi)} dm_i + \vec{v}_{Ci} \times \vec{\omega}_i \int_{(Vi)} \vec{\rho}_i dm_i + \frac{1}{2} \int_{(Vi)} (\vec{\omega}_i \times \vec{\rho}_i)(\vec{\omega}_i \times \vec{\rho}_i) dm_i , \tag{7.95}$$

koji se, prema (7.5) i (7.7) (u izrazu (7.7) : $A \equiv C$) dalje transformiše na sledeći način:

$$E_{ki} = \frac{1}{2} m_i v_{Ci}^2 + \frac{1}{2} \int_{(Vi)} (\vec{\omega}_i \times \vec{\rho}_i)(\vec{\omega}_i \times \vec{\rho}_i) dm_i \tag{7.96}$$

Kako je

$$(\vec{\omega}_i \times \vec{\rho}_i)\cdot(\vec{\omega}_i \times \vec{\rho}_i) = (\vec{\omega}_i \times \vec{\rho}_i)\cdot\{\vec{\omega}_i \times \vec{\rho}_i\} = -(\omega_i)\left[\rho_i^d\right]^2\{\omega_i\}, \tag{7.97}$$

sledi da se (7.96) može napisati u formi:

$$E_{ki} = \frac{1}{2}m_i v_{Ci}^2 + \frac{1}{2}(\omega_i)[J_{ci}]\{\omega_i\}, \tag{7.98}$$

gde je $[J_{Ci}]$ tenzor inercije segmenta (V_i).Uzimajući u obzir poznate relacije (vidi(4.27) i (4.63)):

$$\vec{v}_{Ci} = \sum_{\alpha=1}^{n} \vec{T}_{\alpha(i)}\dot{q}^{\alpha}, \tag{4.27}$$

$$\vec{\omega}_i = \sum_{\alpha=1}^{n} \vec{\Omega}_{\alpha(i)}\dot{q}^{\alpha}, \tag{4.63}$$

izraz (7.98) dobija oblik (7.99)

$$E_{ki} = \frac{1}{2}m_i\left(\sum_{\alpha=1}^{n}\vec{T}_{\alpha(i)}\dot{q}^{\alpha}\right)\cdot\left(\sum_{\beta=1}^{n}\vec{T}_{\beta(i)}\dot{q}^{\beta}\right) + \frac{1}{2}\left(\sum_{\alpha=1}^{n}\vec{\Omega}_{\alpha(i)}\dot{q}^{\alpha}\right)\cdot[J_{Ci}]\cdot\left(\sum_{\beta=1}^{n}\vec{\Omega}_{\beta(i)}\dot{q}^{\beta}\right)$$

ili

$$E_{ki} = \frac{1}{2}\sum_{\alpha=1}^{n}\sum_{\beta=1}^{n} a_{\alpha\beta(i)}\dot{q}^{\alpha}\dot{q}^{\beta}, \tag{7.100}$$

gde je

$$a_{\alpha\beta(i)} = m_i\vec{T}_{\alpha(i)}\vec{T}_{\beta(i)} + (\vec{\Omega}_{\alpha(i)})[J_{Ci}]\{\vec{\Omega}_{\beta(i)}\}. \tag{7.101}$$

Kinetička energija robotskog sistema iznosi

$$E_k = \sum_{i=1}^{n} E_{ki}, \tag{7.102}$$

ili

$$E_k = \frac{1}{2}\sum_{\alpha=1}^{n}\sum_{\beta=1}^{n}\sum_{i=1}^{n} a_{\alpha\beta(i)}\dot{q}^{\alpha}\dot{q}^{\beta}. \tag{7.103}$$

Očigledno, kinetička energija razmatranog sistema (a to je slučaj za mehaničke sisteme koji su podvrgnuti skleronomnim vezama) predstavlja homogenu kvadratnu formu generalisanih brzina $\dot{q}^1, \dot{q}^2, \ldots, \dot{q}^n$. Prema (7.92) i (7.94) sledi

$$\begin{aligned} &E_k(\sum_{i=1}^{n}(\dot{q}^i)^2 \neq 0) > 0, \\ &E_k(\sum_{i=1}^{n}(\dot{q}^i)^2 = 0) = 0, \end{aligned} \tag{7.104}$$

odakle proističe da je (7.103) pozitivno definitna kvadratna forma generalisanih brzina. Izraz (7.103) može da se napiše i u obliku

$$E_k = \frac{1}{2}\sum_{\alpha=1}^{n}\sum_{\beta=1}^{n} a_{\alpha\beta}\dot{q}^{\alpha}\dot{q}^{\beta} \tag{7.105}$$

gde je (vidi (7.103) i (7.105)

$$a_{\alpha\beta} = \sum_{i=1}^{n} m_i \left(\vec{T}_{\alpha(i)}\right)\left\{\vec{T}_{\beta(i)}\right\} + \sum_{i=1}^{n} \left(\vec{\Omega}_{\alpha(i)}\right)\left[J_{Ci}\right]\left\{\vec{\Omega}_{\beta(i)}\right\}, \tag{7.106}$$

pri čemu je očigledno

$$a_{\alpha\beta}(q^1, q^2, ..., q^n) = a_{\beta\alpha}(q^1, q^2, ..., q^n), \tag{7.107}$$

Izraz (7.106) može da se napiše i u obliku

$$a_{\alpha\beta} = \sum_{i=1}^{n} m_i \frac{\partial \vec{r}_{Ci}}{\partial q^{\alpha}} \cdot \frac{\partial \vec{r}_{Ci}}{\partial q^{\beta}} + \sum_{i=1}^{n} \int_{(Vi)} \frac{\partial \vec{\rho}_i}{\partial q^{\alpha}} \cdot \frac{\partial \vec{\rho}_i}{\partial q^{\beta}} dm_i, \tag{7.108}$$

pri čemu su uzete u obzir poznate relacije

$$\frac{\partial \vec{r}_{Ci}}{\partial q^{\alpha}} = \vec{T}_{\alpha(i),} \frac{\partial \vec{\rho}_i}{\partial q^{\alpha}} = \vec{\Omega}_{\alpha(i)} \times \vec{\rho}_i, \left[J_{Ci}\right] = -\int_{(Vi)} \left[\rho_i^d\right]^2 dm_i \tag{7.109}$$

Koeficijenti $a_{\alpha\beta}$ kvadratne forme (7.105) po generalisanim brzinama nazivaju se kovarijantnim• koordinatama osnovnog matričkog tenzora a matrica $\left[a_{\alpha\beta}\right] \in R^{n\times n}$, naziva se *osnovni metrički tenzor*. Primetimo da izraz (7.107) može da se napiše i u obliku (vidi (4.11) i (4.21))

$$a_{\alpha\beta} = \sum_{i=\underline{\alpha\beta}}^{n} m_i \left(\vec{T}_{\alpha(i)}\right)\left\{\vec{T}_{\beta(i)}\right\} + \sum_{i=\underline{\alpha\beta}}^{n} \bar{\xi}_{\alpha}\bar{\xi}_{\beta} \left(\vec{e}_{\alpha(i)}\right)\left[J_{Ci}\right]\left\{\vec{e}_{\beta(i)}\right\}, \tag{7.110}$$

8. Diferencijalne jednačine kretanja robotskog sistema

8.1 Kovarijantni oblik diferencijalnih kretanja robotskog sistema

Razmatramo robotski sistem u obliku otvorenog kinematičkog lanca bez grananja koji je sačinjen od segemenata $[V_1], [V_2], ..., [V_n]$, jednačine veza

• *Izraz preuzet iz tenzorskog računa-vidi, na primer, ( )*

kojima je sistem podvrgnut (veze su holonomne, skleronomne i idealne) omogućavaju, kako je ranije rečeno, da se odabere sistem nezavisnih generalisanih koordinata $(q^1, q^2, ..., q^n)$ koji omogućava da se kinetička energija (7.103) napiše u konačnoj formi u funkciji nezavisnih generalisanih koordinata i njihovih izvoda po vremenu (ako je u pitanju zatvoreni kinematički lanac, dopunske veze u opštem slučaju ne mogu da se dovedu u konačnoj formi na relacije koje izražavaju eksplicitnu zavisnost zavisnih koordinata u funkciji nezavisnih generalisanih koordinata). U tom slučaju diferencijalne jednačine kretanja mogu biti predstavljene u obliku *Langranževih jednačina druge vrste* (izraženih samo u funkciji nezavisnih generalisanih koordinata $(q^1, q^2, ..., q^n)$ i njihovih prvih i drugih izvoda po vremenu), tj., u obliku

$$\frac{d}{dt}\left(\frac{\partial E_k}{\partial \dot{q}^\gamma}\right) - \frac{\partial E_k}{\partial q^\gamma} = Q_\gamma, \ \gamma = 1,2,...,n, \tag{8.1}$$

gde je Q_γ -generalisana sila sistema aktivnih, sila koje deluju na razmatrani robotski sistem, koja odgovara nezavisnoj generalisanoj koordinati q^γ, $(\gamma = 1,2,...,n)$. Kako je (vidi (7.103))

$$\frac{\partial E_k}{\partial \dot{q}^\gamma} = \frac{1}{2}\sum_{\beta=1}^{n} a_{\gamma\beta}\dot{q}^\beta + \frac{1}{2}\sum_{\alpha=1}^{n} a_{\alpha\gamma}\dot{q}^\alpha \tag{8.2}$$

sledi (prema (7.107)):

$$\frac{\partial E_k}{\partial \dot{q}^\gamma} = \sum_{\alpha=1}^{n} a_{\alpha\gamma}\dot{q}^\alpha . \tag{8.3}$$

Dalje je

$$\frac{d}{dt}\left(\frac{\partial E_k}{\partial \dot{q}^\gamma}\right) = \sum_{\alpha=1}^{n} a_{\alpha\gamma}\ddot{q}^\alpha + \sum_{\alpha=1}^{n}\sum_{\beta=1}^{n} \frac{\partial a_{\alpha\gamma}}{\partial q^\beta}\dot{q}^\alpha \dot{q}^\beta , \tag{8.4}$$

Očigledno je da poslednji izraz može da se napiše i u obliku

$$\frac{d}{dt}\left(\frac{\partial E_k}{\partial \dot{q}^\gamma}\right) = \sum_{\alpha=1}^{n} a_{\alpha\gamma}\ddot{q}^\alpha + \frac{1}{2}\sum_{\alpha=1}^{n}\sum_{\beta=1}^{n} \left(\frac{\partial a_{\alpha\gamma}}{\partial q^\beta} + \frac{\partial a_{\beta\gamma}}{\partial q^\alpha}\right)\dot{q}^\alpha \dot{q}^\beta , \tag{8.5}$$

Parcijalni izvod kinetičke enrgije po nezavisnoj generalisanoj koordinati q^γ iznosi

$$\frac{\partial E_k}{\partial q^\gamma} = \frac{1}{2}\sum_{\alpha=1}^{n}\sum_{\beta=1}^{n} \frac{\partial a_{\alpha\gamma}}{\partial q^\beta}\dot{q}^\alpha \dot{q}^\beta , \tag{8.6}$$

tako da jednačine (8.1) dobijaju oblik

$$\sum_{\alpha=1}^{n} a_{\alpha\gamma}\ddot{q}^\alpha + \sum_{\alpha=1}^{n}\sum_{\beta=1}^{n} \Gamma_{\alpha\beta,\gamma}\dot{q}^\alpha \dot{q}^\beta = Q_\gamma \ . \tag{8.7}$$

Ovakav oblik diferencijalnih jednačina kretanja naziva se *kovarijantnim.*U poslednjem izrazu koeficijenti homogene kvadratne forme po generalisanim brzinama iznose

$$\Gamma_{\alpha\beta,\gamma} = \frac{1}{2}\left(\frac{\partial a_{\beta\gamma}}{\partial q^{\alpha}} + \frac{\partial a_{\gamma_{\alpha}}}{\partial q^{\beta}} - \frac{\partial a_{\alpha\beta}}{\partial q^{\gamma}}\right), \tag{8.8}$$

i nazivaju se Kristofelovi simboli prve vrste. Ovi simboli imaju očiglednu osobinu

$$\Gamma_{\alpha\beta,\gamma} = \Gamma_{\beta\alpha,\gamma}, \quad \alpha, \beta, \gamma = 1,2,\ldots,n\ ., \tag{8.9}$$

Ostale osobine Kristofelovih simbola, koje se odnose na sistem u obliku otvorenog kinematičkog lanca, biće navedene u kasnijem izlaganju.Prema (7.108) dobijamo

$$\begin{aligned} \frac{\partial a_{\beta\gamma}}{\partial q^{\alpha}} &= \sum_{i=1}^{n} m_i \left(\frac{\partial^2 \vec{r}_{Ci}}{\partial q^{\alpha}\partial q^{\beta}} \cdot \frac{\partial \vec{r}_{Ci}}{\partial q^{\gamma}} + \frac{\partial \vec{r}_{Ci}}{\partial q^{\beta}} \cdot \frac{\partial^2 \vec{r}_{Ci}}{\partial q^{\alpha}\partial q^{\gamma}}\right) + \\ &+ \sum_{i=1}^{n} \int_{(Vi)} \left(\frac{\partial^2 \vec{\rho}_i}{\partial q^{\alpha}\partial q^{\beta}} \cdot \frac{\partial \vec{\rho}_i}{\partial q^{\gamma}} + \frac{\partial \vec{\rho}_i}{\partial q^{\beta}} \cdot \frac{\partial^2 \vec{\rho}_i}{\partial q^{\alpha}\partial q^{\gamma}}\right) dm_i \end{aligned}, \tag{8.10}$$

odakle sledi (vidi (8.8))

$$\Gamma_{\alpha\beta,\gamma} = \frac{1}{2}\left[\begin{aligned} &\sum_{i=1}^{n} m_i \left(\begin{aligned} &\frac{\partial^2 \vec{r}_{Ci}}{\partial q^{\alpha}\partial q^{\beta}} \cdot \frac{\partial \vec{r}_{Ci}}{\partial q^{\gamma}} + \frac{\partial \vec{r}_{Ci}}{\partial q^{\beta}} \cdot \frac{\partial^2 \vec{r}_{Ci}}{\partial q^{\alpha}\partial q^{\gamma}} + \frac{\partial^2 \vec{r}_{Ci}}{\partial q^{\beta}\partial q^{\gamma}} \cdot \frac{\partial \vec{r}_{Ci}}{\partial q^{\alpha}} \\ &+ \frac{\partial \vec{r}_{Ci}}{\partial q^{\gamma}} \cdot \frac{\partial^2 \vec{r}_{Ci}}{\partial q^{\beta}\partial q^{\alpha}} - \frac{\partial^2 \vec{r}_{Ci}}{\partial q^{\alpha}\partial q^{\gamma}} \cdot \frac{\partial \vec{r}_{Ci}}{\partial q^{\beta}} - \frac{\partial \vec{r}_{Ci}}{\partial q^{\alpha}} \cdot \frac{\partial^2 \vec{r}_{Ci}}{\partial q^{\beta}\partial q^{\gamma}} \end{aligned}\right) + \\ &+ \sum_{i=1}^{n} \int_{(Vi)} \left(\begin{aligned} &\frac{\partial^2 \vec{\rho}_i}{\partial q^{\alpha}\partial q^{\beta}} \cdot \frac{\partial \vec{\rho}_i}{\partial q^{\gamma}} + \frac{\partial \vec{\rho}_i}{\partial q^{\beta}} \cdot \frac{\partial^2 \vec{\rho}_i}{\partial q^{\alpha}\partial q^{\gamma}} + \frac{\partial^2 \vec{\rho}_i}{\partial q^{\gamma}\partial q^{\beta}} \cdot \frac{\partial \vec{\rho}_i}{\partial q^{\alpha}} \\ &+ \frac{\partial \vec{\rho}_i}{\partial q^{\gamma}} \cdot \frac{\partial^2 \vec{\rho}_i}{\partial q^{\alpha}\partial q^{\beta}} - \frac{\partial^2 \vec{\rho}_i}{\partial q^{\alpha}\partial q^{\gamma}} \cdot \frac{\partial \vec{\rho}_i}{\partial q^{\beta}} - \frac{\partial \vec{\rho}_i}{\partial q^{\alpha}} \cdot \frac{\partial^2 \vec{\rho}_i}{\partial q^{\beta}\partial q^{\gamma}} \end{aligned}\right) dm_i \end{aligned}\right], \tag{8.11}$$

i

$$\Gamma_{\alpha\beta,\gamma} = \sum_{i=1}^{n} m_i \frac{\partial^2 \vec{r}_{Ci}}{\partial q^{\alpha}\partial q^{\beta}} \cdot \frac{\partial \vec{r}_{Ci}}{\partial q^{\gamma}} + + \sum_{i=1}^{n} \int_{(Vi)} \frac{\partial^2 \vec{\rho}_i}{\partial q^{\alpha}\partial q^{\beta}} \cdot \frac{\partial \vec{\rho}_i}{\partial q^{\gamma}} dm_i \tag{8.12}$$

Poslednja relacija prema (7.73), (7.84) i (4.35) dobija oblik

$$\begin{aligned} \Gamma_{\alpha\beta,\gamma} &= \sum_{i=1}^{n} m_i \bar{\xi}_{\underline{\alpha\beta}} \left(\vec{e}_{\underline{\alpha\beta}} \times \vec{T}_{\overline{\alpha\beta}(i)}\right) \vec{T}_{\gamma(i)} + \\ &+ \sum_{i=1}^{n} \int_{(Vi)} \vec{\Omega}_{\underline{\alpha\beta}(i)} \times \left(\vec{\Omega}_{\overline{\alpha\beta}(i)} \times \vec{\rho}_i\right) \cdot \left(\vec{\Omega}_{\gamma(i)} \times \vec{\rho}_i\right) dm_i \end{aligned}, \tag{8.13}$$

koji se, uzimajući u obzir relaciju

$$\begin{aligned}&\vec{\Omega}_{\underline{\alpha\beta}(i)}\times\left(\vec{\Omega}_{\overline{\alpha\beta}(i)}\times\vec{\rho}_i\right)\left(\vec{\Omega}_{\gamma(i)}\times\vec{\rho}_i\right)=\left(\vec{\Omega}_{\overline{\alpha\beta}(i)}\cdot\vec{\rho}_i\right)\left[\vec{\Omega}_{\overline{\alpha\beta}(i)}\cdot\vec{\Omega}_{\gamma(i)}\times\vec{\rho}_i\right]=\\&=\left(\vec{\Omega}_{\overline{\alpha\beta}(i)}\times\vec{\Omega}_{\gamma(i)}\right)\left\{\vec{\rho}_i\right\}\left(\vec{\rho}_i\right)\left\{\vec{\Omega}_{\underline{\alpha\beta}(i)}\right\}\end{aligned}, \tag{8.14}$$

dalje transformiše u izraz

$$\begin{aligned}\Gamma_{\alpha\beta,\gamma}&=\sum_{i=1}^{n}m_i\bar{\xi}_{\underline{\alpha\beta}}\left(\vec{e}_{\underline{\alpha\beta}}\times\vec{T}_{\overline{\alpha\beta}(i)}\right)\left\{\vec{T}_{\gamma(i)}\right\}+\\&+\sum_{i=1}^{n}\int_{(Vi)}\left(\vec{\Omega}_{\underline{\alpha\beta}(i)}\times\vec{\Omega}_{\gamma(i)}\right)\left[\int_{(Vi)}\left\{\vec{\rho}_i\right\}\left(\vec{\rho}_i\right)dm_i\right]\cdot\left\{\vec{\Omega}_{\underline{\alpha\beta}(i)}\right\}\end{aligned} \tag{8.15}$$

Očigledno je da važi

$$\int_{(Vi)}\left\{\bar{\rho}_i\right\}\left(\bar{\rho}_i\right)dm_i=\int_{(Vi)}\begin{bmatrix}\xi_i^2 & \xi_i\eta_i & \xi_i\zeta_i\\ \eta_i\xi_i & \eta_i^2 & \eta_i\zeta_i\\ \zeta_i\xi_i & \zeta_i\eta_i & \zeta_i^2\end{bmatrix}dm_i\,, \tag{8.16}$$

ili

$$\int_{(Vi)}\left\{\bar{\rho}_i\right\}\left(\bar{\rho}_i\right)dm_i=\begin{bmatrix}\int_{(Vi)}\xi_i^2dm_i & \int_{(Vi)}\xi_i\eta_i dm_i & \int_{(Vi)}\xi_i\zeta_i dm_i\\ \int_{(Vi)}\eta_i\xi_i dm_i & \int_{(Vi)}\eta_i^2 dm_i & \int_{(Vi)}\eta_i\zeta_i dm_i\\ \int_{(Vi)}\zeta_i\xi_i dm_i & \int_{(Vi)}\zeta_i\eta_i dm_i & \int_{(Vi)}\zeta_i^2 dm_i\end{bmatrix}, \tag{8.17}$$

Desna strana poslednje relacije predstavlja planarni tenzor inercije Π_C segmenta $\left[V_i\right]$. Dijagonalni elementi toga tenzora predstavljaju planarne momente inercije segmneta $\left[V_i\right]$ u odnosu na koordinatne ravni $\eta_i C_i\zeta_i$, $\zeta_i C_i\xi_i$, ,$\xi_i C_i\eta_i$.

$$J_{\eta_iC_i\zeta_i}=\int_{(Vi)}\xi_i^2dm_i, J_{\zeta_iC_i\xi_i}=\int_{(Vi)}\eta_i^2dm_i, J_{\xi_iC_i\eta_i}=\int_{(Vi)}\zeta_i^2dm_i\,, \tag{8.18}$$

dok vandijagonalni elementi toga tenzora predstavljaju centrifugalne momente inercije $J_{\eta_i\zeta_i}, J_{\zeta_i\xi_i}, J_{\xi_i\eta_i}$ segmenta $\left[V_i\right]$ sračunate u odnosu na koordinatni sistem $O\xi_i\eta_i\zeta_i$. Nadalje ćemo pomenuti tenzor inercije označavati sa $\left[\Pi_i\right]$ vodeći računa o činjenici da su njegovi elementi (koordinate) određeni u odnosu na $C_i\xi_i\eta_i\zeta_i$:

$$\left[\Pi_i\right] = \begin{bmatrix} J_{\eta i C\zeta i} & J_{\xi_i \eta i} & J_{\xi_i \zeta i} \\ J_{\eta i \xi i} & J_{\zeta i C i \xi i} & J_{\eta i \zeta i} \\ J_{\zeta_i \xi i} & J_{\zeta i \eta_i} & J_{\xi i C i \eta i} \end{bmatrix} \tag{8.19}$$

Uvođenjem (8.19) Kristiofelovi simboli prve vrste (8.15) mogu da se napišu u obliku

$$\begin{aligned} \Gamma_{\alpha\beta,\gamma} &= \sum_{i=1}^{n} m_i \bar{\xi}_{\underline{\alpha\beta}} \left(\vec{e}_{\underline{\alpha\beta}} \times \vec{T}_{\overline{\alpha\beta}(i)}\right) \cdot \left\{\vec{T}_{\gamma(i)}\right\} + \\ &+ \sum_{i=1}^{n} \left(\vec{\Omega}_{\overline{\alpha\beta}(i)} \times \vec{\Omega}_{\gamma(i)}\right) \left[\Pi_i\right] \left\{\vec{\Omega}_{\overline{\alpha\beta}(i)}\right\} \end{aligned} \tag{8.20}$$

Kako je navedeno ranije (vidi(8.9)) Kristofelovi simboli prve vrste zadovoljavaju uslov

$$\Gamma_{\alpha\beta,\gamma} = \Gamma_{\beta\alpha,\gamma}, \quad \alpha, \beta, \gamma = 1,2,\ldots,n \,., \tag{8.21}$$

Međutim, zadovoljavaju i uslov

$$\Gamma_{\underline{\alpha\beta}\overline{\alpha\beta},\gamma} = -\Gamma_{\underline{\alpha\beta}\gamma,\overline{\alpha\beta},}\,, \tag{8.22}$$

i prema (8.9)

$$\Gamma_{\overline{\alpha\beta}\underline{\alpha\beta},\gamma} = -\Gamma_{\gamma\underline{\alpha\beta},\overline{\alpha\beta},} \tag{8.23}$$

odakle sledi i

$$\Gamma_{\overline{\alpha\beta}\underline{\alpha\beta},\overline{\alpha\beta}} = -\Gamma_{\overline{\alpha\beta}\underline{\alpha\beta},\overline{\alpha\beta}} = 0 \tag{8.24}$$

Primetimo da izraz (8.20) može da se napiše i u formi

$$\begin{aligned} \Gamma_{\alpha\beta,\gamma} &= \sum_{i=\overline{\alpha\beta\gamma}}^{n} m_i \bar{\xi}_{\underline{\alpha\beta}} \left(\vec{e}_{\underline{\alpha\beta}} \times \vec{T}_{\overline{\alpha\beta}(i)}\right) \cdot \left\{\vec{T}_{\gamma(i)}\right\} + \\ &+ \sum_{i=\overline{\alpha\beta\gamma}}^{n} \bar{\xi}_{\underline{\alpha\beta}} \bar{\xi}_{\overline{\alpha\beta}} \bar{\xi}_{\gamma} \left(\vec{e}_{\overline{\alpha\beta}} \times \vec{e}_{\gamma)}\right) \left[\Pi_i\right] \left\{\vec{e}_{\overline{\alpha\beta}}\right\} \end{aligned} \tag{8.25}$$

Izraz (8.7) u potpunosti je određen izrazima (8.25) i (7.110). Za potpuno određivanje kovarijantnog oblika diferencijalnih jednačina kretanja (8.7) robotskog sistema potrebno je odrediti i generalisane sile $Q_\alpha \, (\alpha = 1,2,..,n)$ koje deluju na robotski sistem.