Sa ispitnih rokova i kolokvijuma

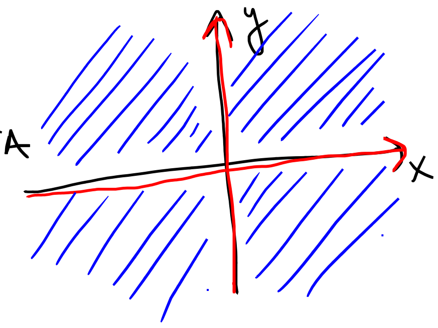
(I kolokv. 2023.)

1. Naći lokalne ekstremne vrednosti funkcije

$$z(x,y)=\frac{4x^4+y^4+8}{xy}.$$

Domen: $x\neq 0,\ y\neq 0$

Domen funkcije su sve tačke osim koordinatnih osa



Parcijalni izvodi:

$$z'_x=\frac{16x^3\cdot xy-(4x^4+y^4+8)\cdot y}{x^2y^2}=\frac{16x^4y-4x^4y-y^5-8y}{x^2y^2}=\frac{12x^4y-y^5-8y}{x^2y^2}=\frac{12x^4-y^4-8}{x^2y}$$

$$z'_y=\frac{4y^3\cdot xy-(4x^4+y^4+8)\cdot x}{x^2y^2}=\frac{4xy^4-4x^5-xy^4-8x}{x^2y^2}=\frac{3xy^4-4x^5-8x}{x^2y^2}=\frac{3y^4-4x^4-8}{xy^2}$$

Stacionarne tačke se dobijaju kao rešenje sistema

$$\left.\begin{matrix} z'_x=0 \\ z'_y=0 \end{matrix}\right\}\Rightarrow \begin{matrix} 12x^4-y^4-8=0 \\ 3y^4-4x^4-8=0 \ /\cdot(-1)\ + \end{matrix}$$

$$12x^4-3y^4+4x^4-y^4=0$$

$$16x^4-4y^4=0$$

$$4x^4-y^4=0$$

$$(2x^2+y^2)\cdot(2x^2-y^2)=0$$

I $\quad 2x^2+y^2=0$ $\quad$ ili $\quad$ II $\quad 2x^2-y^2=0$

$$\underbrace{y^2}_{>0}=\underbrace{-2x^2}_{<0}$$

nema rešenja

$$y^2=2x^2$$

$$12x^4-4x^4-8=0$$

$$8x^4-8=0$$

$$x^4=1$$

$$x=\pm 1$$

$x=1$, $y^2=2$, $y=\pm\sqrt{2}$ $\qquad$ $x=-1$, $y^2=2$

$M_1(1,\sqrt{2})$ $\ M_2(1,-\sqrt{2})$ $\qquad$ $M_3(-1,\sqrt{2})$ $\ M_4(-1,-\sqrt{2})$

Drugi parcijalni izvodi

$$z''_{xx} = (z'_x)'_x = \frac{48x^3 \cdot x^2y - (12x^4 - y^4 - 8)\cdot 2xy}{x^4y^2} = \frac{\overbrace{48x^5y - 24x^5y}^{24x^5y} + 2xy^5 + 16xy}{x^4y^2} \overset{:xy}{=} \frac{24x^4 + 2y^4 + 16}{x^3y}$$

$$z''_{xy} = (z'_x)'_y = \frac{-4y^3 \cdot x^2y - (12x^4 - y^4 - 8)\cdot x^2}{x^4y^2} = \frac{-4x^2y^4 - 12x^6 + x^2y^4 + 8x^2}{x^4y^2} \overset{:x^2}{=} \frac{-3y^4 - 12x^4 + 8}{x^2y^2}$$

$$z''_{yy} = (z'_y)'_y = \frac{12y^3 \cdot xy^2 - (3y^4 - 4x^4 - 8)\cdot 2xy}{x^2y^4} = \frac{\overbrace{12xy^5 - 6xy^5}^{6xy^5} + 8x^5y + 16xy}{x^2y^4} \overset{:xy}{=} \frac{6y^4 + 8x^4 + 16}{xy^3}$$

Tačka $M_1(1, \sqrt{2})$

$$z''_{xx}(M_1) = \frac{24 + 8 + 16}{\sqrt{2}} = \frac{48}{\sqrt{2}} = A$$

$$z''_{xy}(M_1) = \frac{-12 - 12 + 8}{2} = -8 = B$$

$$z''_{yy}(M_1) = \frac{12 + 8 + 16}{2\sqrt{2}} = \frac{36}{2\sqrt{2}} = \frac{18}{\sqrt{2}} = C$$

$$\Delta = B^2 - AC = 64 - \frac{48 \cdot \cancel{18}^{9}}{\cancel{\sqrt{2}} \cdot \cancel{\sqrt{2}}} < 0$$

$A > 0$

Tačka M_1 je tačka lokalnog minimuma

$$z_{min}(1, \sqrt{2}) = \frac{4 + 4 + 8}{\sqrt{2}} = \frac{16}{\sqrt{2}} = 8\sqrt{2}$$

Tačka $M_2(1, -\sqrt{2})$

$$z''_{xx} = -\frac{48}{\sqrt{2}}$$

$$z''_{xy} = -8$$

$$z''_{yy} = -\frac{18}{\sqrt{2}}$$

$\Delta < 0$

$A < 0$

Tačka M_2 je tačka lokalnog maksimuma

$$z_{max}(1, -\sqrt{2}) = \frac{4 + 4 + 8}{-\sqrt{2}} = -8\sqrt{2}$$

Tačka $M_3(-1, \sqrt{2})$

$$z''_{xx} = -\frac{48}{\sqrt{2}}$$

$$z''_{xy} = -8$$

$$z''_{yy} = -\frac{18}{\sqrt{2}}$$

$\Delta < 0$

$A < 0$

M_3 lok. max

$$z_{max}(-1, \sqrt{2}) = \frac{4 + 4 + 8}{-\sqrt{2}} = -8\sqrt{2}$$

Tačka $M_4(-1, -\sqrt{2})$

$$z''_{xx} = \frac{48}{\sqrt{2}}$$

$$z''_{xy} = -8$$

$$z''_{yy} = \frac{18}{\sqrt{2}}$$

$\Delta < 0$

$A > 0$

M_4 je lok. min

$$z_{min}(-1, -\sqrt{2}) = 8\sqrt{2}$$

2. (I kolok. 2024)

Naći lokalne ekstremne vrednosti funkcije $z(x,y)=\frac{x^4+y^2+4}{xy}$.

Slično kao prethodni (za vežbu)

Tačke $M_1(-\sqrt{2},2\sqrt{2})$ i $M_2(\sqrt{2},-2\sqrt{2})$ su tačke lokalnog maksimuma

$$z_{max}=-4$$

Tačke $M_3(-\sqrt{2},-2\sqrt{2})$ i $M_4(-\sqrt{2},2\sqrt{2})$ su tačke lokalnog minimuma

$$z_{min}=4$$

3. (Jul 2024.)

Naći ekstremne vrednosti funkcije $z(x,y)=(y-x)^3+x^4+y^4$

Domen: $\mathbb{R}^2$

$$z'_x=3(y-x)^2\cdot(-1)+4x^3=-3(y-x)^2+4x^3$$

$$z'_y=3(y-x)^2+4y^3$$

$$\left.\begin{array}{l} z'_x=0 \\ z'_y=0 \end{array}\right\} \Rightarrow \left.\begin{array}{l} -3(y-x)^2+4x^3=0 \\ 3(y-x)^2+4y^3=0 \end{array}\right\} +$$

$$4x^3+4y^3=0$$

$$(x+y)\cdot\underbrace{(x^2-xy+y^2)}_{\neq 0}=0$$

$$x+y=0$$

$$\boxed{y=-x}$$

$$-12x^2+4x^3=0$$

$$4x^2\cdot(-3+x)=0$$

$x=0$ ili $-3+x=0$

$y=0$ $x=3$

$y=-3$

$M_1(0,0)$ $M_2(3,-3)$

$z''_{xx} = 6(y-x) + 12x^2$

$z''_{xy} = -6(y-x)$

$z''_{yy} = 6(y-x) + 12y^2$

za tačku $M_1(0,0)$

$z''_{xx} = z''_{xy} = z''_{yy} = 0$

Tačka M_1 nije tačka ekstremuma.

$z(0,0) = 0$

$z(0,\varepsilon) = \varepsilon^3 + \varepsilon^4 > 0$

$z(\varepsilon,0) = -\varepsilon^3 + \varepsilon^4 < 0$

z menja znak u okolini tačke $(0,0)$, pa je tačka $M_1(0,0)$ sedlasta tačka.

za tačku $M_2(3,-3)$

$z''_{xx}(M_2) = -36 + 108 = 72 = A$

$z''_{xy} = 36 = B$

$z''_{yy} = -36 + 108 = 72 = C$

$\Delta = B^2 - AC = 36 \cdot 36 - 72 \cdot 72 < 0$

$A = 72 > 0$

Tačka M_2 je tačka lokalnog minimuma

$z_{min}(3,-3) = -216 + 81 + 81 = -54$

4. (OKTOBAR 2024.)

Naći lokalne ekstremne vrednosti funkcije

$$z(x,y) = (y-x)^3 + \frac{1}{2}(x^4+y^4)$$

$\boxed{\begin{array}{c}(0,0) \text{ sedlasta tačka} \\ M(6,-6) \text{ lok. min.} \\ z_{min} = -432\end{array}}$

slično kao prethodni, uradite za vežbu.

5. (JANUAR 2024.)

Naći lokalne ekstremne vrednosti funkcije

$$z = \frac{1+x+y}{\sqrt{2+x^2+y^2}}$$

Domen: $2+x^2+y^2>0$ $\quad D=\mathbb{R}^2$

$x,y\in\mathbb{R}$

$$z'_x = \frac{\sqrt{2+x^2+y^2}-(1+x+y)\cdot\frac{1}{\cancel{2}\sqrt{2+x^2+y^2}}\cdot\cancel{2}x}{2+x^2+y^2} = \frac{2+x^2+y^2-x-x^2-xy}{(2+x^2+y^2)\sqrt{2+x^2+y^2}} =$$

$$= \frac{2+y^2-x-xy}{(2+x^2+y^2)^{\frac{3}{2}}}$$

$$z'_y = \frac{\sqrt{2+x^2+y^2}-(1+x+y)\cdot\frac{1}{\cancel{2}\sqrt{2+x^2+y^2}}\cdot\cancel{2}y}{2+x^2+y^2} = \frac{2+x^2-y-xy}{(2+x^2+y^2)^{\frac{3}{2}}}$$

$$\left.\begin{array}{l} z'_x = 0 \\ z'_y = 0 \end{array}\right\} \Rightarrow \begin{array}{l} 2 + y^2 - x - xy = 0 \;/\cdot(-1) \\ 2 + x^2 - y - xy = 0 \end{array}$$

$$x^2 - y^2 + x - y = 0$$

$$(x-y)(x+y) + (x-y) = 0$$

$$(x-y)\cdot(x+y+1) = 0$$

I $\quad x - y = 0$

$$x = y$$

$$2 + x^2 - x - x^2 = 0$$

$$x = 2$$

$$y = 2$$

$$M_1(2,2)$$

II $\quad x + y + 1 = 0$

$$y = -x - 1$$

$$2 + (-x-1)^2 - x - x(-x-1) = 0$$

$$2 + x^2 + 2x + 1 - x + x^2 + x = 0$$

$$2x^2 + 2x + 3 = 0$$

$$x_{1/2} = \frac{-2 \pm \sqrt{4 - 24}}{4} \notin \mathbb{R}$$

$$z''_{xx} = \frac{(-1-y)\cdot(2+x^2+y^2)^{\frac{3}{2}} - \frac{3}{\cancel{2}}\cdot(2+y^2-x-xy)\cdot(2+x^2+y^2)^{\frac{1}{2}}\cdot\cancel{2}x}{(2+x^2+y^2)^3}$$

$$= \frac{(2+x^2+y^2)^{\frac{1}{2}}\cdot\left((-1-y)\cdot(2+x^2+y^2) - 3x\cdot(2+y^2-x-xy)\right)}{(2+x^2+y^2)^3}$$

$$z''_{xy} = \frac{(2y-x)\cdot(2+x^2+y^2)^{\frac{3}{2}} - \frac{3}{2}(2+y^2-x-xy)\cdot(2+x^2+y^2)^{\frac{1}{2}}\cdot 2y}{(2+x^2+y^2)^3} =$$

$$= \frac{(2+x^2+y^2)^{\frac{1}{2}}\left((2y-x)\cdot(2+x^2+y^2) - 3y(2+y^2-x-xy)\right)}{(2+x^2+y^2)^3}$$

$$z''_{yy} = \frac{(-1-x)\cdot(2+x^2+y^2)^{\frac{3}{2}} - \frac{3}{2}(2+x^2-y-xy)\cdot(2+x^2+y^2)^{\frac{1}{2}}\cdot 2y}{(2+x^2+y^2)^3} =$$

$$= \frac{(2+x^2+y^2)^{\frac{1}{2}}\left((-1-x)\cdot(2+x^2+y^2) - 3y(2+x^2-y-xy)\right)}{(2+x^2+y^2)^3}$$

$M(2,2)$

$$z''_{xx}(M)=\frac{\sqrt{10}\cdot(-3\cdot 10-6\cdot 0)}{10^3}=-\frac{3\sqrt{10}}{100}=A$$

$$z''_{xy}=\frac{\sqrt{10}\cdot(2\cdot 10-6\cdot 0)}{10^3}=\frac{2\sqrt{10}}{100}=B$$

$$z''_{yy}=\frac{\sqrt{10}\cdot(-3\cdot 10-6\cdot 0)}{10^3}=-\frac{3\sqrt{10}}{100}=C$$

$$\Delta=B^2-AC=\frac{4\cdot 10}{100^2}-\frac{9\cdot 10}{100^2}<0$$

$A<0$

Tačka $M(2,2)$ je tačka lokalnog maksimuma

$$z_{max}(2,2)=\frac{1+2+2}{\sqrt{10}}=\frac{5}{\sqrt{10}}=\frac{\sqrt{10}}{2}$$

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6. (JUN 2013.)

Naći Tejlorov polinom drugog stepena za funkciju $z=z(x,y)$ datu implicitno jednačinom $e^{xy}+xz-2yz+z^2=4$ u tački $M(0,1)$, $z(0,1)<0$.

Tejlorov polinom 2. stepena u tački $M(a,b)$ je:

$$T_2(M)=z(a,b)+z'_x(a,b)(x-a)+z'_y(a,b)(y-b)+\frac{1}{2!}\Big(z''_{xx}(a,b)(x-a)^2+2z''_{xy}(a,b)(x-a)(y-b)$$
$$+z''_{yy}(a,b)(y-b)^2\Big)$$

Prvo ćemo naći vrednost $z(0,1)$

$$e^0+0-2z+z^2=4 \quad (\text{ovde je } z \text{ zapravo } z(0,1))$$

$$z^2-2z-3=0$$

$z_1=3 \quad z_2=-1$ Kako imamo uslov $z(0,1)<0 \Rightarrow \boxed{z(0,1)=-1}$

Tražimo parcijalne izvode I i II reda

Neka je $F(x,y,z(x,y))=e^{xy}+xz-2yz+z^2-4$

$$\left.\begin{aligned} F'_x&=ye^{xy}+z \\ F'_y&=xe^{xy}-2z \\ F'_z&=x-2y+2z \end{aligned}\right\} \Rightarrow \begin{aligned} z'_x&=-\frac{F'_x}{F_z}=-\frac{ye^{xy}+z}{x-2y+2z} \Rightarrow \boxed{z'_x(0,1)=-\frac{1-1}{-2-2}=0} \\ z'_y&=-\frac{F'_y}{F_z}=-\frac{xe^{xy}-2z}{x-2y+2z} \Rightarrow \boxed{z'_y(0,1)=-\frac{2}{-4}=\frac{1}{2}} \end{aligned}$$

$$z''_{xx}=(z'_x)'_x=-\frac{(ye^{xy}+z)'_x\cdot(x-2y+2z)-(ye^{xy}+z)\cdot(x-2y+2z)'_x}{(x-2y+2z)^2}=$$

$$=-\frac{(y^2e^{xy}+z'_x)\cdot(x-2y+2z)-(ye^{xy}+z)\cdot(1-2z'_x)}{(x-2y+2z)^2}$$

$$z''_{xy}=(z'_x)'_y=-\frac{(e^{xy}+xye^{xy}+z'_y)\cdot(x-2y+2z)-(ye^{xy}+z)\cdot(-2+2z'_y)}{(x-2y+2z)^2}$$

$$z''_{yy}=(z'_y)'_y=-\frac{(x^2e^{xy}-2z'_y)\cdot(x-2y+2z)-(xe^{xy}-2z)\cdot(-2+2z'_y)}{(x-2y+2z)^2}$$

$$z''_{xx}(0,1)=-\frac{1\cdot(-4)-(1-1)\cdot(1-0)}{16}=\frac{1}{4}$$

$$z''_{xy}(0,1)=-\frac{(1+\frac{1}{2})\cdot(-4)-0}{16}=\frac{3}{8}$$

$$z''_{yy}(0,1)=-\frac{4-2\cdot(-1)}{16}=-\frac{3}{8}$$

$$T_2(0,1)=-1+0\cdot(x-0)+\frac{1}{2}(y-1)+\frac{1}{2!}\left(\frac{1}{4}(x-0)^2+2\cdot\frac{3}{8}\cdot(x-0)(y-1)-\frac{3}{8}(y-1)^2\right)$$

$$\boxed{T_2(0,1)=-1+\frac{1}{2}(y-1)+\frac{1}{8}x^2+\frac{3}{8}x\cdot(y-1)-\frac{3}{16}(y-1)^2}$$

INTEGRALI

1. (II kolokvijum 2022.)

$$\int \frac{x-3}{(x+1)^2(x^2+2x+2)}\,dx = ?$$

$$\frac{x-3}{(x+1)^2(x^2+2x+2)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{Cx+D}{x^2+2x+2} \quad \Big/ \cdot (x+1)^2(x^2+2x+2)$$

$$x-3 = A(x+1)(x^2+2x+2) + B(x^2+2x+2) + (Cx+D)\underbrace{(x+1)^2}_{x^2+2x+1}$$

za $x=-1$ $\quad \boxed{-4 = B}$

$$x-3 = A\cdot(x^3+3x^2+4x+2) + B(x^2+2x+2) + Cx^3+2Cx^2+Cx + Dx^2+2Dx+D$$

uz x^3: $\quad 0 = A + C \quad \Rightarrow \boxed{C = -A}$

x^2: $\quad 0 = 3A + B + 2C + D$

x: $\quad 1 = 4A + 2B + C + 2D$

1: $\quad -3 = 2A + 2B + D$

$$\left.\begin{aligned} D &= 4 - A \\ 2D &= 9 - 3A \\ D &= 5 - 2A \end{aligned}\right\} \Rightarrow \boxed{\begin{aligned} A &= 1 \\ D &= 3 \\ C &= -1 \\ B &= -4 \end{aligned}}$$

$$I = \int \frac{1}{x+1}dx - \int \frac{4}{(x+1)^2}dx + \int \frac{-x+3}{x^2+2x+2}dx$$

$$= \ln|x+1| + \frac{4}{x+1} - \frac{1}{2}\int \frac{(2x+2)+4}{x^2+2x+2}dx =$$

$$= \ln|x+1| + \frac{4}{x+1} - \frac{1}{2}\int \frac{(2x+2)dx}{x^2+2x+2} - 2\int \frac{1}{x^2+2x+2}dx =$$

$$= \ln|x+1| + \frac{4}{x+1} - \frac{1}{2}\int \frac{d(x^2+2x+2)}{x^2+2x+2} - 2\cdot\int \frac{1}{(x+1)^2+1}dx =$$

$$\boxed{= \ln|x+1| + \frac{4}{x+1} - \frac{1}{2}\ln(x^2+2x+2) - 2\,\mathrm{arctg}(x+1) + C}$$

$\boxed{\text{Sept. 2024.}}$

2. $\int (x^2-2x)e^{-\frac{x}{2}}dx = ?$

$u = x^2-2x \qquad dv = e^{-\frac{x}{2}}dx$

$du = (2x-2)dx \qquad v = \int e^{-\frac{x}{2}}dx = \left[\begin{array}{l} t=-\frac{x}{2} \\ dt = -\frac{dx}{2} \\ -2dt = dx \end{array}\right] = -2\int e^t dt = -2e^t = -2e^{-\frac{x}{2}}$

$I = uv - \int v\,du = -2(x^2-2x)\cdot e^{-\frac{x}{2}} + 2\int (2x-2)e^{-\frac{x}{2}}dx =$

$= -2(x^2-2x)e^{-\frac{x}{2}} + 4\underbrace{\int (x-1)e^{-\frac{x}{2}}dx}_{*} = *$

$*\quad u = x-1 \qquad dv = e^{-\frac{x}{2}}dx$

$du = dx \qquad v = -2e^{-\frac{x}{2}}$

$u\cdot v \qquad - \int v\,du$

$= -2(x^2-2x)e^{-\frac{x}{2}} + 4\cdot\left(-2(x-1)e^{-\frac{x}{2}} + 2\underbrace{\int e^{-\frac{x}{2}}dx}_{-2e^{-\frac{x}{2}}}\right) =$

$= -2(x^2-2x)e^{-\frac{x}{2}} - 8(x-1)e^{-\frac{x}{2}} - 16e^{-\frac{x}{2}} + C =$

$= -2e^{-\frac{x}{2}}(x^2-2x+4x-4+8) + C$

$$\boxed{I = -2e^{-\frac{x}{2}}(x^2+2x+4) + C}$$

(II kolok. 2024.)

3. $\int x \operatorname{arctg}(2x)\,dx = ?$

$u = \operatorname{arctg}(2x) \qquad dv = x\,dx$

$du = \frac{1}{1+(2x)^2} \cdot 2 = \frac{2}{1+4x^2} \qquad v = \int x\,dx = \frac{x^2}{2}$

$I = uv - \int v\,du = \frac{x^2}{2}\operatorname{arctg}(2x) - \int \frac{x^2}{1+4x^2}\,dx =$

$= \frac{x^2}{2}\operatorname{arctg}(2x) - \frac{1}{4}\int \frac{(4x^2+1)-1}{1+4x^2}\,dx =$

smena $t = 2x$

$= \frac{x^2}{2}\operatorname{arctg}(2x) - \frac{1}{4}\int \cancel{\frac{4x^2+1}{1+4x^2}}\,dx + \frac{1}{4}\int \frac{1}{1+\underbrace{4x^2}_{(2x)^2}}\,dx =$

$= \frac{x^2}{2}\operatorname{arctg}(2x) - \frac{1}{4}x + \frac{1}{8}\operatorname{arctg}(2x) + C$

4. JUN 2023.

$$\int \frac{x\ln x}{\sqrt{x^2-2}}\,dx = ?$$

$u = \ln x \qquad dv = \dfrac{x}{\sqrt{x^2-2}}\,dx$

$du = \dfrac{1}{x}\,dx$

$$v = \int \frac{x}{\sqrt{x^2-2}}\,dx = \left[\begin{array}{l} t^2 = x^2-2 \\ 2t\,dt = 2x\,dx \\ t\,dt = x\,dx \end{array}\right] = \int \frac{t\,dt}{t} = \int dt = t = \sqrt{x^2-2}$$

$$I = \sqrt{x^2-2}\,\ln x - \boxed{\int \frac{\sqrt{x^2-2}}{x}\,dx}\ ^{*}$$

$$^{*}\int \frac{\sqrt{x^2-2}}{x}\,dx = \int \frac{x\sqrt{x^2-2}}{x^2}\,dx = \left[\begin{array}{l} t^2 = x^2-2 \\ t\,dt = x\,dx \end{array}\right] = \int \frac{t^2\,dt}{t^2+2} = \int \frac{(t^2+2)-2}{t^2+2}\,dt =$$

$$= \int \cancel{\frac{t^2+2}{t^2+2}}\,dt - 2\int \frac{1}{t^2+2}\,dt = t - 2\cdot\frac{1}{\sqrt{2}}\operatorname{arctg}\frac{t}{\sqrt{2}} = \boxed{\sqrt{x^2-2} - \sqrt{2}\operatorname{arctg}\frac{\sqrt{x^2-2}}{\sqrt{2}}}$$

$$I = \sqrt{x^2-2}\,\ln x - \sqrt{x^2-2} + \sqrt{2}\operatorname{arctg}\frac{\sqrt{x^2-2}}{\sqrt{2}} + C$$

$$\boxed{I = \sqrt{x^2-2}\,(\ln x - 1) + \sqrt{2}\operatorname{arctg}\frac{\sqrt{x^2-2}}{\sqrt{2}} + C}$$

5. (JAN. 2024.)

$$\int x^2\sqrt{1-x^2}\,dx = \left[\begin{array}{l} x=\sin t \\ dx=\cos t\,dt \end{array}\right] = \int \sin^2 t\cdot\underbrace{\sqrt{1-\sin^2 t}}_{\cos t}\cdot\cos t\,dt =$$

$$=\int \sin^2 t\cdot\cos^2 t\,dt = \int \frac{1-\cos 2t}{2}\cdot\frac{1+\cos 2t}{2}\,dt = \frac{1}{4}\int(1-\cos^2 2t)\,dt$$

$$=\frac{1}{4}\int dt - \frac{1}{4}\int\frac{1+\cos 4t}{2}\,dt = \frac{1}{4}t - \frac{1}{8}\int dt - \frac{1}{8}\underbrace{\int\cos 4t\,dt}_{\text{smena } 4t=s}$$

$$=\frac{1}{4}t - \frac{1}{8}t - \frac{1}{8}\cdot\frac{1}{4}\sin 4t + C = \frac{1}{8}t - \frac{1}{32}\sin 4t + C =$$

$$=\frac{1}{8}\arcsin x - \frac{1}{32}\sin 4\cdot(\arcsin x) + C$$

6. (II kolok. 2022.)

$$\int \frac{3x}{\cos^4 x}\,dx = ?$$

$$I = 3\cdot\int \frac{x}{\cos^4 x}\,dx = \begin{bmatrix} u = x & dv = \int \frac{1}{\cos^4 x}\,dx \\ du = dx & v = \overset{*}{\int} \frac{1}{\cos^4 x}\,dx \end{bmatrix}$$

$$*\,V = \overbrace{\int \frac{1}{\cos^4 x}\,dx}^{I'} = \frac{\operatorname{tg} x}{\cos^2 x} - 2\int \frac{\sin x}{\cos^2 x}\cdot \operatorname{tg} x\,dx = \frac{\operatorname{tg} x}{\cos^2 x} - 2\int \frac{\sin^2 x}{\cos^4 x}\,dx =$$

$$u' = \frac{1}{\cos^2 x} \qquad dv' = \frac{1}{\cos^2 x}\,dx$$

$$du' = \frac{2\sin x}{\cos^2 x}\,dx \qquad v' = \operatorname{tg} x$$

$$= \frac{\operatorname{tg} x}{\cos^2 x} - 2\cdot\int \frac{1-\cos^2 x}{\cos^4 x}\,dx = \frac{\operatorname{tg} x}{\cos^2 x} - 2\underbrace{\int \frac{1}{\cos^4 x}\,dx}_{I'} + 2\int \frac{\cancel{\cos^2 x}}{\cos^{\cancel{4}\,2} x}\,dx =$$

$$= \frac{\operatorname{tg} x}{\cos^2 x} - 2I' + 2\operatorname{tg} x \Rightarrow 3I' = \operatorname{tg} x\left(\frac{1}{\cos^2 x} + 2\right)$$

$$\boxed{V = I' = \frac{\operatorname{tg} x}{3}\left(\frac{1}{\cos^2 x} + 2\right)}$$

$$I = 3\cdot\left(\frac{x\operatorname{tg} x}{3}\left(\frac{1}{\cos^2 x} + 2\right) - \int \frac{\operatorname{tg} x}{3}\left(\frac{1}{\cos^2 x} + 2\right)dx\right) =$$

smena: $\cos x = t$ smena: $\cos x = t$

$$= x\operatorname{tg} x\left(\frac{1}{\cos^2 x} + 2\right) - \int \frac{\sin x}{\cos^3 x}\,dx - 2\cdot\int \frac{\sin x}{\cos x}\,dx =$$

$$\boxed{= x\operatorname{tg} x\left(\frac{1}{\cos^2 x} + 2\right) - \frac{1}{2}\cdot\frac{1}{\cos^2 x} + 2\ln|\cos x| + C}$$

7. (II kolok. 2024.)

$$\int \frac{x^2}{\sqrt{2+2x-x^2}}\,dx = ?$$

metod Ostrogradskog

$$\int \frac{x^2}{\sqrt{2+2x-x^2}}\,dx = (Ax+B)\cdot\sqrt{2+2x-x^2} + \lambda\int\frac{1}{\sqrt{2+2x-x^2}}\,dx \quad \Big/'$$

$$\frac{x^2}{\sqrt{2+2x-x^2}} = A\cdot\sqrt{2+2x-x^2} + (Ax+B)\cdot\frac{1\cdot(\cancel{2}-\cancel{2}x)}{\cancel{2}\sqrt{2+2x-x^2}} + \frac{\lambda}{\sqrt{2+2x-x^2}} \quad \Big/\cdot\sqrt{2+2x-x^2}$$

$$x^2 = \underbrace{A\cdot(2+2x-x^2)}_{2A+2Ax-Ax^2} + \underbrace{(Ax+B)(1-x)}_{Ax+B-Ax^2-Bx} + \lambda$$

$$x^2 = -2Ax^2 + (3A-B)x + 2A+B+\lambda$$

$$\left.\begin{aligned} x^2:&\quad 1 = -2A \\ x:&\quad 0 = 3A-B \\ 1:&\quad 0 = 2A+B+\lambda \end{aligned}\right\} \Rightarrow \boxed{A=-\frac{1}{2}} \quad \boxed{B=-\frac{3}{2}}$$

$$\lambda = -2A-B$$

$$\lambda = +1+\frac{3}{2}$$

$$\boxed{\lambda=\frac{5}{2}}$$

$$I = \left(-\frac{1}{2}x-\frac{3}{2}\right)\cdot\sqrt{2+2x-x^2} + \frac{5}{2}\underbrace{\int\frac{1}{\sqrt{2+2x-x^2}}\,dx}$$

$$2+2x-x^2 = 3-(1-x)^2$$

$$\int\frac{1}{\sqrt{2+2x-x^2}}\,dx = \int\frac{1}{\sqrt{3-(1-x)^2}}\,dx = \left|\begin{matrix} t=1-x \\ dt=-dx \end{matrix}\right| = -\int\frac{1}{\sqrt{3-t^2}}\,dt = -\arcsin\frac{t}{\sqrt{3}}$$

$$= -\arcsin\frac{1-x}{\sqrt{3}}$$

Dakle,

$$\boxed{I = \frac{-x-3}{2}\sqrt{2+2x-x^2} - \frac{5}{2}\arcsin\frac{1-x}{\sqrt{3}} + C}$$