Смена променљиве

$\cdot \int f(x)dx = \left\{ \begin{matrix} x=\varphi(t) \\ dx=\varphi'(t)dt \end{matrix} \right\} = \int f(\varphi(t))\,\varphi'(t)dt = \int f(\varphi(t))\,d\varphi(t)$

1. $\int (2x+3)^{1000}dx = \left\{ \begin{matrix} 2x+3=t \\ 2dx=dt \end{matrix} \right\} = \int t^{1000}\frac{dt}{2} = \frac{1}{2}\frac{t^{1001}}{1001}+C = \frac{(2x+3)^{1001}}{2002}+C$

вратити смену

2. начин: $\int (2x+3)^{1000}dx = \int (2x+3)^{1000}\cdot\frac{1}{2}d(2x+3) = \frac{1}{2}\frac{(2x+3)^{1001}}{1001}+C$

провера: $\left(\frac{1}{2}\frac{(2x+3)^{1001}}{1001}+C\right)' = \frac{1}{2\cdot 1001}\cdot 1001(2x+3)^{1000}\cdot 2 = (2x+3)^{1000}$

2. $\int \frac{dx}{3x-2} = \left\{ \begin{matrix} 3x-2=t \\ 3dx=dt \end{matrix} \right\} = \int \frac{dt/3}{t} = \frac{1}{3}\ln|t|+C = \frac{1}{3}\ln|3x-2|+C$

2. начин: $\int \frac{dx}{3x-2} = \int \frac{\frac{1}{3}d(3x-2)}{3x-2} = \frac{1}{3}\ln|3x-2|+C$

напомена: $\int \frac{dx}{2-3x} = \left\{ \begin{matrix} 2-3x=t \\ -3dx=dt \end{matrix} \right\}$

3. $\int \cos 2x\,dx = \left\{ \begin{matrix} 2x=t \\ 2dx=dt \end{matrix} \right\} = \int \cos t\cdot\frac{1}{2}dt = \frac{1}{2}\sin t+C = \frac{1}{2}\sin 2x+C$

2. начин: $\int \cos 2x\,dx = \int \cos 2x\cdot\frac{1}{2}d(2x) = \frac{1}{2}\sin 2x+C$

4. $\int \frac{3x^2-2}{x^3-2x}dx = \left\{ \begin{matrix} x^3-2x=t \\ (3x^2-2)dx=dt \end{matrix} \right\} = \int \frac{dt}{t} = \ln|t|+C = \ln|x^3-2x|+C$

2. начин: $\int \frac{3x^2-2}{x^3-2x}dx = \int \frac{d(x^3-2x)}{x^3-2x} = \ln|x^3-2x|+C$

5. $\int \frac{x\,dx}{\sqrt[3]{x^2-1}} = \left\{ \begin{matrix} x^2-1=t \\ 2x\,dx=dt \end{matrix} \right\} = \int \frac{dt/2}{\sqrt[3]{t}} = \frac{1}{2}\int t^{-1/3}dt = \frac{1}{2}\frac{t^{2/3}}{2/3}+C =$

$= \frac{3}{4}(x^2-1)^{2/3}+C$

2. начин: $\int \frac{x\,dx}{\sqrt[3]{x^2-1}} = \left\{ \begin{matrix} x^2-1=t^3 \\ 2x\,dx=3t^2dt \end{matrix} \right\} = \int \frac{\frac{3t^2dt}{2}}{\sqrt[3]{t^3}} = \frac{3}{2}\int t\,dt =$

$= \frac{3}{2}\frac{t^2}{2}+C = \frac{3}{4}(x^2-1)^{2/3}+C$

6. $\underbrace{\int \frac{2x - \sqrt{\arcsin x}}{\sqrt{1-x^2}}\,dx}_{I} = \underbrace{\int \frac{2x\,dx}{\sqrt{1-x^2}}}_{I_1} - \underbrace{\int \frac{\sqrt{\arcsin x}}{\sqrt{1-x^2}}\,dx}_{I_2}$

$I_1 = \left\{ \begin{array}{l} 1-x^2 = t^2 \\ -2x\,dx = 2t\,dt \end{array} \right\} = \int \frac{-2t\,dt}{t} = -2t + C_1 = -2\sqrt{1-x^2} + C_1$

$I_2 = \left\{ \begin{array}{l} \arcsin x = t \\ \frac{dx}{\sqrt{1-x^2}} = dt \end{array} \right\} = \int \sqrt{t}\,dt = \frac{t^{3/2}}{3/2} + C_2 = \frac{2}{3}(\arcsin x)^{3/2} + C_2$

$I = I_1 - I_2 = -2\sqrt{1-x^2} - \frac{2}{3}(\arcsin x)^{3/2} + C$

7. $\int \frac{dx}{x \ln x} = \left\{ \begin{array}{l} \ln x = t \\ \frac{dx}{x} = dt \end{array} \right\} = \int \frac{dt}{t} = \ln|t| + C = \ln|\ln x| + C$

8. $\int \frac{dx}{x \ln x \cdot \ln(\ln x)} = \left\{ \begin{array}{l} \ln x = t \\ \frac{dx}{x} = dt \end{array} \right\} = \int \frac{dt}{t \ln t} = \left\{ \begin{array}{l} \ln t = s \\ \frac{dt}{t} = ds \end{array} \right\} = \int \frac{ds}{s} =$

$= \ln|s| + C = \ln|\ln t| + C = \ln|\ln(\ln x)| + C$

2. начин: $\int \frac{dx}{x \ln x \cdot \ln(\ln x)} = \left\{ \begin{array}{l} \ln(\ln x) = t \\ \frac{dx}{(\ln x)\cdot x} = dt \end{array} \right\} = \int \frac{dt}{t} = \ln|t| + C =$

$= \ln|\ln(\ln x)| + C$

9. $\int \frac{e^x dx}{1+e^{2x}} = \left\{ \begin{array}{l} e^x = t \\ e^x dx = dt \end{array} \right\} = \int \frac{dt}{1+t^2} = \operatorname{arctg} t + C = \operatorname{arctg} e^x + C$

10. $\int \frac{e^{3x}-1}{e^x-1}\,dx = \int \frac{\cancel{(e^x-1)}(e^{2x}+e^x+1)}{\cancel{e^x-1}}\,dx = \frac{1}{2}e^{2x} + e^x + x + C$

11. $\int x(1-x)^{10}\,dx = \left\{ \begin{array}{l} 1-x = t \\ -dx = dt \end{array} \right\} = \int (1-t)\,t^{10}(-dt) = \int (t^{11} - t^{10})\,dt =$

$= \frac{t^{12}}{12} - \frac{t^{11}}{11} + C = \frac{(1-x)^{12}}{12} - \frac{(1-x)^{11}}{11} + C$

$$x^3 = \frac{2-t^3}{5}$$

12. $$\int \underbrace{x^5}_{= x^3 \cdot x^2} (2-5x^3)^{2/3} dx = \left\{ \begin{array}{l} 2-5x^3 = t^3 \\ -15x^2 dx = 3t^2 dt \end{array} \right\} = \int \frac{2-t^3}{5} \cdot (t^3)^{2/3} \left(-\frac{t^2 dt}{5}\right) =$$

$$x^2 dx = -\frac{t^2 dt}{5}$$

$$= \frac{1}{25} \int (t^3-2) \cdot t^2 \cdot t^2 dt = \frac{1}{25} \int (t^7 - 2t^4) dt = \frac{1}{25}\left(\frac{t^8}{8} - 2\frac{t^5}{5}\right) + C$$

$$= \frac{1}{200} (2-5x^3)^{8/3} - \frac{2}{125} (2-5x^3)^{5/3} + C$$

13. $$\int x\sqrt{x+2}\, dx = \left\{ \begin{array}{l} x+2 = t^2 \\ dx = 2t\,dt \end{array} \right\} = \int (t^2-2) \cdot t \cdot 2t\, dt =$$

$$= 2\int (t^4 - 2t^2) dt = 2\left(\frac{t^5}{5} - 2\frac{t^3}{3}\right) + C = \frac{2}{5}(x+2)^{5/2} - \frac{4}{3}(x+2)^{3/2} + C$$

14. $$\int \frac{dx}{\sqrt{x(1-x)}} = \int \frac{dx}{\sqrt{x}\sqrt{1-x}} = \left\{ \begin{array}{l} \sqrt{x} = t \\ \frac{dx}{2\sqrt{x}} = dt \end{array} \right\} = \int \frac{2dt}{\sqrt{1-t^2}} = 2\arcsin t + C =$$

$$= 2\arcsin\sqrt{t} + C$$

15. $$\int \frac{dx}{\sqrt{x+2} - \sqrt{x+1}} = \int \frac{dx}{\sqrt{x+2} - \sqrt{x+1}} \cdot \frac{\sqrt{x+2} + \sqrt{x+1}}{\sqrt{x+2} + \sqrt{x+1}} = \int \frac{\sqrt{x+2} + \sqrt{x+1}}{\cancel{x}+2-\cancel{x}-1} dx$$

$$= \int \left((x+2)^{1/2} + (x+1)^{1/2}\right) dx = \frac{(x+2)^{3/2}}{3/2} + \frac{(x+1)^{3/2}}{3/2} + C =$$

$$= \frac{2}{3}\left((x+2)^{3/2} + (x+1)^{3/2}\right) + C$$

16. $$\int \sqrt{\frac{1-x}{1+x}}\, dx = \int \sqrt{\frac{1-x}{1+x}} \cdot \sqrt{\frac{1-x}{1-x}}\, dx = \int \frac{1-x}{\sqrt{1-x^2}} dx = \int \frac{dx}{\sqrt{1-x^2}} - \int \frac{x\,dx}{\sqrt{1-x^2}} =$$

$$\cancel{= \arcsin x - \int d\sqrt{1-x^2} \quad \int d\sqrt{1-x} = \arcsin x}$$

$$= \int \frac{dx}{\sqrt{1-x^2}} - \int d(-\sqrt{1-x^2}) = \arcsin x + \sqrt{1-x^2} + C$$

17. $\int \frac{dx}{x\sqrt{x^2-1}} = \int \frac{x\,dx}{x^2\sqrt{x^2-1}} = \left| \begin{matrix} x^2-1 = t^2 \\ 2x\,dx = 2t\,dt \end{matrix} \right| = \int \frac{t\,dt}{(t^2+1)\cancel{t}} = \operatorname{arctg} t + C =$

$= \boxed{\operatorname{arctg}\sqrt{x^2-1} + C}$

2. начин: $\int \frac{dx}{x\sqrt{x^2-1}} = \int \frac{dx}{x\sqrt{x^2(1-\frac{1}{x^2})}} = \int \frac{dx}{x^2\sqrt{1-(\frac{1}{x})^2}} = \left| \begin{matrix} \frac{1}{x} = t \\ -\frac{dx}{x^2} = dt \end{matrix} \right| =$

$= \int \frac{-dt}{\sqrt{1-t^2}} = -\arcsin t + C = \boxed{-\arcsin\frac{1}{x} + C}$

провера: $(\operatorname{arctg}\sqrt{x^2-1} + C)' = \frac{1}{1+(\sqrt{x^2-1})^2} \cdot \frac{1}{\cancel{2}\sqrt{x^2-1}} \cdot \cancel{2}x =$

$= \frac{1}{\cancel{1}+x^{\cancel{2}}-\cancel{1}} \, \frac{\cancel{x}}{\sqrt{x^2-1}} = \frac{1}{x\sqrt{x^2-1}}$

$(-\arcsin\frac{1}{x} + C)' = -\frac{1}{\sqrt{1-(\frac{1}{x})^2}} \cdot \left(-\frac{1}{x^2}\right) = \frac{1}{x^2\sqrt{\frac{x^2-1}{\cancel{x^2}}}}$

$= \frac{1}{x\sqrt{x^2-1}}$

18. $\int \frac{dx}{\sqrt{x}(1+\sqrt[3]{x})} = \left| \begin{matrix} x = t^6 \\ dx = 6t^5 dt \end{matrix} \right| = \int \frac{6t^5 dt}{t^3(1+t^2)} = 6\int \frac{t^2 dt}{1+t^2} =$

$= 6\int \frac{t^2+1-1}{1+t^2} dt = 6\left(\int dt - \int \frac{dt}{1+t^2}\right) = 6(t - \operatorname{arctg} t) + C =$

$= 6(\sqrt[6]{x} - \operatorname{arctg}\sqrt[6]{x}) + C.$

19. $\int \sqrt{1-x^2}\,dx = \left| \begin{matrix} x = \sin t \\ dx = \cos t \end{matrix} \right| = \int \underbrace{\underbrace{\sqrt{1-\sin^2 t}}_{=\cos^2 t}}_{=\cos t} \cdot \cos t\,dt = \int \cos^2 t\,dt =$

$= \int \frac{1+\cos 2t}{2} dt = \frac{1}{2}(t + \frac{1}{2}\sin 2t) + C = \frac{1}{2}\arcsin x + \frac{1}{4}\sin(2\arcsin x) + C.$

~~20.~~ $\cancel{\int \frac{dx}{x\sqrt{1-x^2}} \left| \begin{matrix} x = \sin t \\ dx = \cos t\,dt \end{matrix} \right| = \int \frac{\cos t\,dt}{}}$

20. $$\int \frac{dx}{x^2+a^2} = \int \frac{dx}{a^2\left(\frac{x^2}{a^2}+1\right)} = \frac{1}{a^2}\int \frac{dx}{\left(\frac{x}{a}\right)^2+1} = \left\{ \begin{array}{l} \frac{x}{a} = t \\ \frac{dx}{a} = dt \end{array} \right\} = \frac{1}{a^2}\int \frac{a\,dt}{t^2+1} =$$

$$= \frac{1}{a} \operatorname{arctg} t + c = \frac{1}{a} \operatorname{arctg} \frac{x}{a} + C$$

21. $$\int \frac{dx}{\sqrt{a^2-x^2}} = \int \frac{dx}{\sqrt{a^2\left(1-\frac{x^2}{a^2}\right)}} = \int \frac{dx}{a\sqrt{1-\left(\frac{x}{a}\right)^2}} = \left\{ \begin{array}{l} \frac{x}{a} = t \\ \frac{dx}{a} = dt \end{array} \right\} =$$

$(a>0)$

$$= \int \frac{dt}{\sqrt{1-t^2}} = \arcsin t + C = \arcsin \frac{x}{a} + C$$

22. $$\int \sqrt{a^2-x^2}\,dx = \int \sqrt{a^2\left(1-\frac{x^2}{a^2}\right)}\,dx = \int a\sqrt{1-\left(\frac{x}{a}\right)^2}\,dx = \left\{ \begin{array}{l} \frac{x}{a} = \sin t \\ \frac{dx}{a} = \cos t\,dt \end{array} \right\} =$$

$(a>0)$

$$= \int a \underbrace{\sqrt{1-\sin^2 t}}_{=\sqrt{\cos^2 t} = \cos t} \cdot a\cos t\,dt = a^2 \int \cos^2 t\,dt \overset{\boxed{19}}{=} \dots = a^2\left(\frac{1}{2}t + \frac{1}{4}\sin 2t\right) + C =$$

$$= a^2\left(\frac{1}{2}\arcsin\frac{x}{a} + \frac{1}{4}\sin\left(2\cdot\arcsin\frac{x}{a}\right)\right) + C$$

23. $$\int \operatorname{tg} 3x\,dx = \int \frac{\sin 3x}{\cos 3x}\,dx = \left\{ \begin{array}{l} \cos 3x = t \\ -3\sin 3x\,dx = dt \end{array} \right\} = \int \frac{-\frac{1}{3}dt}{t} =$$

$$= -\frac{1}{3}\ln|t| + C = -\frac{1}{3}\ln|\cos 3x| + C$$

24. $$\int \sin 3x \cdot \sin 6x\,dx = \int \frac{1}{2}(\cos 3x - \cos 9x)\,dx = \frac{1}{6}\sin 3x - \frac{1}{18}\sin 9x + C$$

25. $$\int \underbrace{\cos x \cdot \cos 2x}\cdot \cos 5x\,dx = \int \frac{1}{2}\underbrace{(\cos 3x + \cos x)}\cos 5x\,dx =$$

$$= \frac{1}{2}\int (\cos 3x \cdot \cos 5x + \cos x \cdot \cos 5x)\,dx =$$

$$= \frac{1}{2} \int \left(\frac{1}{2}(\cos 8x + \cos 2x) + \frac{1}{2}(\cos 6x + \cos 4x)\right) dx =$$

$$= \frac{1}{4} \int (\cos 2x + \cos 4x + \cos 6x + \cos 8x)\, dx =$$

$$= \frac{1}{8} \sin 2x + \frac{1}{16} \sin 4x + \frac{1}{24} \sin 6x + \frac{1}{32} \sin 8x + C$$

26. $$\int \frac{\cos x\, dx}{\sqrt{2+\cos 2x}} = \left\{ \sqrt{2+\cos 2x} = \sqrt{2+\cos^2 x - \sin^2 x} = \sqrt{2+1-\sin^2 x - \sin^2 x} = \sqrt{3-2\sin^2 x} \right\} =$$

$$= \int \frac{\cos x\, dx}{\sqrt{3-2\sin^2 x}} = \left\{ \begin{array}{l} \sin x = t \\ \cos x\, dx = dt \end{array} \right\} = \int \frac{dt}{\sqrt{3-2t^2}} = \int \frac{dt}{\sqrt{3\left(1-\frac{2t^2}{3}\right)}} =$$

$$= \frac{1}{\sqrt{3}} \int \frac{dt}{\sqrt{1-\left(t\sqrt{\frac{2}{3}}\right)^2}} = \left\{ \begin{array}{l} t\sqrt{\frac{2}{3}} = s \\ \sqrt{\frac{2}{3}}\, dt = ds \end{array} \right\} = \cancel{\frac{1}{\sqrt{3}}} \int \frac{\sqrt{\frac{3}{2}}\, ds}{\sqrt{1-s^2}} =$$

$$= \frac{1}{\sqrt{2}} \arcsin s + C = \frac{1}{\sqrt{2}} \arcsin\left(t\sqrt{\frac{2}{3}}\right) + C = \frac{1}{\sqrt{2}} \arcsin\left(\sqrt{\frac{2}{3}} \sin x\right) + C.$$

27. $$\int \sin^4 x \cdot \sin 2x\, dx = \int (\sin^2 x)^2 \cdot \sin 2x\, dx = \left\{ \begin{array}{l} \sin^2 x = t \\ \underbrace{2\sin x \cos x}_{\sin 2x}\, dx = dt \end{array} \right\}$$

$$= \int t^2\, dt = \frac{t^3}{3} + C = \frac{1}{3} \sin^6 x\, dx$$

2. начин: $$\int \sin^4 x \cdot \sin 2x\, dx = \int \sin^4 x \cdot 2\sin x \cos x\, dx =$$

$$= 2\int \sin^5 x \cos x\, dx = \left\{ \begin{array}{l} \sin x = t \\ \cos x\, dx = dt \end{array} \right\} = 2\int t^5\, dt =$$

$$= 2\frac{t^6}{6} + C = \frac{1}{3} \sin^6 x + C$$

28. $$\int \sin^2 x\, dx = \int \frac{1-\cos 2x}{2}\, dx = \frac{1}{2}\left(x - \frac{1}{2}\sin 2x\right) + C = \frac{1}{2}x - \frac{1}{4}\sin 2x + C.$$

29. $\int \sin^4 x\,dx = \int (\sin^2 x)^2 dx = \int \left(\frac{1-\cos 2x}{2}\right)^2 dx = \frac{1}{4}\int (1 - 2\cos 2x + \cos^2 2x)\,dx =$

$= \frac{1}{4}\int \left(1 - 2\cos 2x + \frac{1+\cos 4x}{2}\right)dx = \frac{1}{8}\int (3 - 4\cos 2x + \cos 4x)\,dx =$

$= \frac{3}{8}x - \frac{1}{4}\sin 2x + \frac{1}{32}\cos 4x + C$

30. $\int \sin^5 x\,dx = \int \sin^4 x \cdot \sin x\,dx = \int (\sin^2 x)^2 \sin x\,dx = \int (1-\cos^2 x)^2 \sin x\,dx =$

$= \left\{\begin{matrix} \cos x = t \\ -\sin x\,dx = dt \end{matrix}\right\} = \int (1-t^2)^2(-dt) = -\int (t^4 - 2t^2 + 1)\,dt =$

$= -\frac{t^5}{5} + \frac{2t^3}{3} - t + C = -\frac{\cos^5 x}{5} + \frac{2\cos^3 x}{3} - \cos x + C$

31. $\int \frac{dx}{\sin x} = \int \frac{dx}{2\sin\frac{x}{2}\cos\frac{x}{2}} = \int \frac{dx}{2\cdot\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}\cdot\cos^2\frac{x}{2}} = \int \frac{dx}{2\,tg\frac{x}{2}\cdot\cos^2\frac{x}{2}} =$

$= \left\{\begin{matrix} tg\frac{x}{2} = t \\ \frac{dx}{2\cos^2\frac{x}{2}} = dt \end{matrix}\right\} = \int \frac{dt}{t} = \ln|t| + C = \ln\left|tg\frac{x}{2}\right| + C$

32. $\int \frac{dx}{\cos x} = \int \frac{dx}{\sin(x+\frac{\pi}{2})} = \left\{\begin{matrix} x+\frac{\pi}{2} = t \\ dx = dt \end{matrix}\right\} = \int \frac{dt}{\sin t} = \boxed{31} = \ln\left|tg\frac{t}{2}\right| + C =$

$= \ln\left|tg\frac{x+\pi/2}{2}\right| + C$

33. $\int \frac{x\,dx}{\sin x^2} = \left\{\begin{matrix} x^2 = t \\ 2x\,dx = dt \end{matrix}\right\} = \int \frac{dt/2}{\sin t} = \boxed{31} = \frac{1}{2}\ln\left|tg\frac{t}{2}\right| + C =$

$\cancel{\frac{1}{2}\ln|tg\,x|} = \frac{1}{2}\ln\left|tg\frac{x^2}{2}\right| + C$

34. $\int \frac{dx}{x\sqrt{1-x^2}} = \left\{\begin{matrix} x = \sin t \\ dx = \cos t\,dt \end{matrix}\right\} = \int \frac{\cos t\,dt}{\sin t\,\cancel{\sqrt{1-\sin^2 t}}} = \boxed{31} = \frac{1}{2}\ln\left|tg\frac{t}{2}\right| + C =$

$= \frac{1}{2}\ln\left|tg\frac{\arcsin x}{2}\right| + C$

35. $\int \frac{dx}{1-\cos x} = \int \frac{dx}{2\sin^2 \frac{x}{2}} = \left\{ \begin{array}{l} \frac{x}{2} = t \\ \frac{dx}{2} = dt \end{array} \right\} = \int \frac{dt}{\sin^2 t} = -\operatorname{ctg} t + C =$

$= -\operatorname{ctg} \frac{x}{2} + C$

36. $\int \sqrt{1-\sin 2x}\, dx = \int \sqrt{1-\sin 2x}\, \frac{\sqrt{1+\sin 2x}}{\sqrt{1+\sin 2x}}\, dx = \int \frac{\sqrt{1-\sin^2 2x}}{\sqrt{1+\sin 2x}}\, dx =$

$= \int \frac{\cos 2x\, dx}{\sqrt{1+\sin 2x}} = \left\{ \begin{array}{l} 1+\sin 2x = t^2 \\ \cancel{2}\cos 2x\, dx = \cancel{2}t\,dt \end{array} \right\} = \int \frac{t\,dt}{\cancel{\sqrt{t^2}}} = t + C =$

$= \sqrt{1+\sin 2x} + C.$