Парцијална интеграција

- $\int u\,dv = uv - \int v\,du$

1\. $\int xe^x dx = \left\{ \begin{matrix} u = x, & dv = e^x dx \\ du = dx, & v = e^x \end{matrix} \right\} = xe^x - \int e^x dx = xe^x - e^x + C$

2\. начин: $\int xe^x dx = \int x\, d(e^x) = xe^x - \int e^x dx = xe^x - e^x + C$

2\. $\int x^2 e^{3x} dx = \left\{ \begin{matrix} u = x^2, & dv = e^{3x} dx \\ du = 2x\,dx, & v = \frac{1}{3} e^{3x} \end{matrix} \right\} =$

$= \frac{1}{3} x^2 e^{3x} - \frac{2}{3} \int x e^{3x} dx = \left\{ \begin{matrix} u = x, & dv = e^{3x} dx \\ du = dx, & v = \frac{1}{3} e^{3x} \end{matrix} \right\} =$

$= \frac{1}{3} x^2 e^{3x} - \frac{2}{3} \left(\frac{1}{3} x e^{3x} - \frac{1}{3} \int e^{3x} dx \right) =$

$= \frac{1}{3} x^2 e^{3x} - \frac{2}{9} x e^{3x} + \frac{2}{27} e^{3x} + C = \frac{1}{27} (9x^2 - 6x + 2) e^{3x} + C$

(4) 3. $\int \frac{x\,dx}{\sin^2 x} = \left\{ \begin{matrix} u = x, & dv = \frac{dx}{\sin^2 x} \\ du = dx, & v = -\mathrm{ctg}\,x \end{matrix} \right\} = -x\,\mathrm{ctg}\,x + \int \mathrm{ctg}\,x\, dx =$

$= -x\,\mathrm{ctg}\,x + \cancel{\int \mathrm{ctg}\,x\, d} + \int \frac{\cos x}{\sin x} dx = \left\{ \begin{matrix} \sin x = t \\ \cos x\, dx = dt \end{matrix} \right\} =$

$= -x\,\mathrm{ctg}\,x + \int \frac{dt}{t} = -x\,\mathrm{ctg}\,x + \ln|t| + C = -x\,\mathrm{ctg}\,x + \ln|\sin x| + C$

(3) 4. $\int \frac{x\,dx}{\cos^2 x} = \left\{ \begin{matrix} u = x, & dv = \frac{dx}{\cos^2 x} \\ du = dx, & v = \mathrm{tg}\,x \end{matrix} \right\} = x\,\mathrm{tg}\,x - \int \mathrm{tg}\,x\, dx =$

$= x\,\mathrm{tg}\,x - \int \frac{\sin x}{\cos x} dx = x\,\mathrm{tg}\,x + \int \frac{d(\cos x)}{\cos x} = x\,\mathrm{tg}\,x + \ln|\cos x| + C$

5. $\int \ln x\,dx = \left\{\begin{matrix} u = \ln x, & dv = dx \\ du = \frac{dx}{x}, & v = x \end{matrix}\right\} = x\ln x - \int dx = x\ln x - x + C$

6. $\int \ln^2 x\,dx = \left\{\begin{matrix} u = \ln^2 x, & dv = dx \\ du = 2\ln x \cdot \frac{1}{x}dx, & v = x \end{matrix}\right\} = x\ln^2 x - 2\int \ln x\,dx = \ldots$

$\overset{\boxed{5}}{=} x\ln^2 x - 2x\ln x + 2x + C$

(8) 7. $\int \arcsin x\,dx = \left\{\begin{matrix} u = \arcsin x, & dv = dx \\ du = \frac{dx}{\sqrt{1-x^2}}, & v = x \end{matrix}\right\} = x\arcsin x - \int \frac{x\,dx}{\sqrt{1-x^2}} =$

$= \left\{\begin{matrix} 1-x^2 = t^2 \\ -\cancel{2}x\,dx = \cancel{2}t\,dt \end{matrix}\right\} = x\arcsin x + \int \frac{t\,dt}{\sqrt{t^2}} = x\arcsin x + t + C =$

$= x\arcsin x + \sqrt{1-x^2} + C$

(9) 8. $\int x^2\arccos x\,dx = \left\{\begin{matrix} u = \arccos x, & dv = x^2dx \\ du = -\frac{dx}{\sqrt{1-x^2}}, & v = \frac{x^3}{3} \end{matrix}\right\} =$

$= \frac{x^3\arccos x}{3} + \frac{1}{3}\int \frac{x^3\,dx}{\sqrt{1-x^2}} = \left\{\begin{matrix} 1-x^2 = t^2 \\ -\cancel{2}x\,dx = \cancel{2}t\,dt \end{matrix}\right\} =$

$= \frac{x^3\arccos x}{3} + \frac{1}{3}\int \frac{(1-t^2)(-\cancel{t}dt)}{\cancel{\sqrt{t^2}}} = \frac{1}{3}\left(x^3\arccos x + \int (t^2-1)dt\right)$

$= \frac{1}{3}\left(x^3\arccos x + \frac{t^3}{3} - t\right) + C = \frac{1}{3}\left(x^3\arccos x + \frac{1}{3}(1-x^2)^{3/2} - (1-x^2)^{1/2}\right) + C$

(10) 9. $\int (\arcsin x)^2\,dx = \left\{\begin{matrix} u = (\arcsin x)^2, & dv = dx \\ du = \frac{2\arcsin x\,dx}{\sqrt{1-x^2}}, & v = x \end{matrix}\right\} =$

$= x(\arcsin x)^2 - 2\int \frac{x\arcsin x\,dx}{\sqrt{1-x^2}} = \left\{\begin{matrix} u = \arcsin x, \quad dv = \frac{x\,dx}{\sqrt{1-x^2}} \\ du = \frac{dx}{\sqrt{1-x^2}}, \\ v = \int \frac{x\,dx}{\sqrt{1-x^2}} = \left\{\begin{matrix} 1-x^2 = t^2 \\ -\cancel{2}x\,dx = \cancel{2}t\,dt \end{matrix}\right\} \\ = \int \frac{-t\,dt}{\sqrt{t^2}} = -t = -\sqrt{1-x^2} \end{matrix}\right\} =$

$= x(\arcsin x)^2 + 2\sqrt{1-x^2}\arcsin x - 2\int dx$

$= x(\arcsin x)^2 + 2\sqrt{1-x^2}\arcsin x - 2x + C$

(7) 10. $\int \operatorname{arctg} x \, dx = \left\{ \begin{array}{l} u = \operatorname{arctg} x, \quad dv = dx \\ du = \frac{dx}{1+x^2}, \quad v = x \end{array} \right\} = x \operatorname{arctg} x - \int \frac{x\,dx}{1+x^2} =$

$= \left\{ \begin{array}{l} 1+x^2 = t \\ 2x\,dx = dt \end{array} \right\} = x \operatorname{arctg} x - \int \frac{dt/2}{t} = x \operatorname{arctg} x - \frac{1}{2} \ln|t| + C =$

$= x \operatorname{arctg} x - \frac{1}{2} \ln(1+x^2) + C$

11. $\int \operatorname{arctg} \sqrt{x} \, dx = \left\{ \begin{array}{l} u = \operatorname{arctg} \sqrt{x}, \quad dv = dx \\ du = \frac{1}{1+x} \cdot \frac{1}{2\sqrt{x}} \, dx, \quad v = x \end{array} \right\} =$

$= x \operatorname{arctg} \sqrt{x} - \frac{1}{2} \int \frac{x\,dx}{\sqrt{x}(1+x)} = x \operatorname{arctg} \sqrt{x} - \frac{1}{2} \int \frac{(x+1)-1}{\sqrt{x}(1+x)} \, dx =$

$= x \operatorname{arctg} \sqrt{x} - \frac{1}{2} \int \frac{dx}{\sqrt{x}} + \frac{1}{2} \int \frac{dx}{\sqrt{x}(1+x)} = \left\{ \begin{array}{l} \sqrt{x} = t \\ \frac{dx}{2\sqrt{x}} = dt \end{array} \right\} =$

$= x \operatorname{arctg} \sqrt{x} - \cancel{\frac{1}{2}} \frac{\sqrt{x}}{\cancel{1/2}} + \frac{1}{2} \int \frac{2\,dt}{1+t^2} = x \operatorname{arctg} \sqrt{x} - \sqrt{x} + \operatorname{arctg} t + C =$

$= x \operatorname{arctg} \sqrt{x} - \sqrt{x} + \operatorname{arctg} \sqrt{x} + C.$

12. $\int x^3 \cos 2x \, dx = \left\{ \begin{array}{l} u = x^3, \quad dv = \cos 2x \, dx \\ du = 3x^2 dx, \quad v = \frac{1}{2} \sin 2x \end{array} \right\} =$

$= \frac{1}{2} x^3 \sin 2x - \frac{3}{2} \int x^2 \sin 2x \, dx = \left\{ \begin{array}{l} u = x^2, \quad dv = \sin 2x \, dx \\ du = 2x\,dx, \quad v = -\frac{1}{2} \cos 2x \end{array} \right\} =$

$= \frac{1}{2} x^3 \sin 2x - \frac{3}{2} \left(-\frac{1}{2} x^2 \cos 2x + \int x \cos 2x \, dx \right) = \left\{ \begin{array}{l} u = x, \ dv = \cos 2x \, dx, \\ du = dx, \ v = \frac{1}{2} \sin 2x \end{array} \right\} =$

$= \frac{1}{2} x^3 \sin 2x + \frac{3}{4} x^2 \cos 2x - \frac{3}{2} \left(\frac{1}{2} x \sin 2x - \frac{1}{2} \int \sin 2x \, dx \right) =$

$= \frac{1}{2} x^3 \sin 2x + \frac{3}{4} x^2 \cos 2x - \frac{3}{4} x \sin 2x + \frac{3}{8} \cos 2x + C$

13. $\int x \ln^2 x \, dx = \left\{ \begin{array}{l} u = \ln^2 x, \; dv = x\,dx \\ du = 2\ln x \cdot \frac{1}{x}\,dx, \; v = \frac{x^2}{2} \end{array} \right\} =$

$= \frac{x^2 \ln^2 x}{2} - \int x \ln x \, dx = \left\{ \begin{array}{l} u = \ln x, \; dv = x\,dx \\ du = \frac{dx}{x}, \; v = \frac{x^2}{2} \end{array} \right\} =$

$= \frac{x^2 \ln^2 x}{2} - \frac{x^2 \ln x}{2} + \frac{1}{2}\int x\,dx = \frac{x^2 \ln^2 x}{2} - \frac{x^2 \ln x}{2} + \frac{x^2}{4} + C$

14. $\int x^2 \ln^3 x \, dx = \left\{ \begin{array}{l} u = \ln^3 x, \; dv = x^2 dx \\ du = 3\ln^2 x \cdot \frac{1}{x}\,dx, \; v = \frac{x^3}{3} \end{array} \right\} =$

$= \frac{x^3 \ln^3 x}{3} - \int x^2 \ln^2 x \, dx = \left\{ \begin{array}{l} u = \ln^2 x, \; dv = x^2 dx \\ du = 2\ln x \cdot \frac{1}{x}\,dx, \; v = \frac{x^3}{3} \end{array} \right\} =$

$= \frac{x^3 \ln^3 x}{3} - \frac{x^3 \ln^2 x}{3} + \frac{2}{3}\int x^2 \ln x \, dx = \left\{ \begin{array}{l} u = \ln x, \; dv = x^2 dx \\ du = \frac{dx}{x}, \; v = \frac{x^3}{3} \end{array} \right\} =$

$= \frac{x^3 \ln^3 x}{3} - \frac{x^3 \ln^2 x}{3} + \frac{2}{3}\,\frac{x^3 \ln x}{3} - \frac{2}{9}\int x^2 dx =$

$= \frac{x^3 \ln^3 x}{3} - \frac{x^3 \ln^2 x}{3} + \frac{2x^3 \ln x}{9} - \frac{2}{9}\,\frac{x^3}{3} + C =$

$= \frac{x^3}{27}\left(9\ln^3 x - 9\ln^2 x + 6\ln x - 2\right) + C$

15. $\int x \ln \frac{x+1}{x}\,dx = \left\{ \begin{array}{l} u = \ln \frac{x+1}{x}, \; dv = x\,dx \\ du = \frac{1}{\frac{x+1}{x}}\,\frac{1\cdot x - (x+1)\cdot 1}{x^2}\,dx = \frac{-dx}{x(x+1)}, \; v = \frac{x^2}{2} \end{array} \right\} =$

$= \frac{x^2}{2}\ln\frac{x+1}{x} + \frac{1}{2}\int \frac{x\,dx}{x+1} = \frac{x^2}{2}\ln\frac{x+1}{x} + \frac{1}{2}\int \frac{(x+1) - 1}{x+1}\,dx =$

$= \frac{x^2}{2}\ln\frac{x+1}{x} + \frac{1}{2}x - \frac{1}{2}\ln|x+1| + C$

16. $$\int \frac{x\ln\left(x+\sqrt{x^2+1}\right)}{\sqrt{x^2+1}}\,dx = \left\{\begin{array}{l} u=\ln\left(x+\sqrt{x^2+1}\right), \quad dv=\dfrac{x\,dx}{\sqrt{x^2+1}} \\ du=\dfrac{1+\frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+1}}}{x+\sqrt{x^2+1}}\,dx = \dfrac{\frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}}}{x+\sqrt{x^2+1}}\,dx \\ v=\displaystyle\int\frac{x\,dx}{\sqrt{x^2+1}} = \left\{\begin{array}{l} x^2+1=t^2 \\ \cancel{2}x\,dx=\cancel{2}t\,dt\end{array}\right\} = \int\frac{t\,dt}{\sqrt{t^2}} = \\ = t=\sqrt{x^2+1}\end{array}\right\} =$$

$$= \sqrt{x^2+1}\,\ln\left(x+\sqrt{x^2+1}\right) - \int dx = \sqrt{x^2+1}\,\ln\left(x+\sqrt{x^2+1}\right) - x + C$$

17. $$\int e^x\sin x\,dx = \left\{\begin{array}{l} u=e^x, \quad dv=\sin x\,dx \\ du=e^x dx, \quad v=-\cos x\end{array}\right\} =$$

$$= -e^x\cos x + \int e^x\cos x\,dx = \left\{\begin{array}{l} u=e^x, \quad dv=\cos x\,dx \\ du=e^x dx, \quad v=\sin x\end{array}\right\} =$$

$$= -e^x\cos x + e^x\sin x - \int e^x\sin x\,dx$$

$$2\int e^x\sin x\,dx = (\sin x-\cos x)\cdot e^x + C$$

$$\int e^x\sin x\,dx = \frac{\sin x-\cos x}{2}e^x + C$$

18. $$\underset{(a,b\neq 0)}{\int e^{ax}\sin bx\,dx} = \left\{\begin{array}{l} u=e^{ax}, \quad dv=\sin bx\,dx \\ du=ae^{ax}dx, \quad v=-\frac{1}{b}\cos bx\end{array}\right\} =$$

$$= -\frac{1}{b}e^{ax}\cos bx + \frac{a}{b}\int e^{ax}\cos bx\,dx = \left\{\begin{array}{l} u=e^{ax}, \quad dv=\cos bx\,dx \\ du=ae^{ax}dx, \quad v=\frac{1}{b}\sin bx\end{array}\right\} =$$

$$= -\frac{1}{b}e^{ax}\cos bx + \frac{a}{b}\left(\frac{1}{b}e^{ax}\sin bx - \frac{a}{b}\int e^{ax}\sin bx\,dx\right)$$

$$= -\frac{1}{b}e^{ax}\cos bx + \frac{a}{b^2}e^{ax}\sin bx - \frac{a^2}{b^2}\int e^{ax}\sin bx\,dx$$

$$\left(1+\frac{a^2}{b^2}\right)\int e^{ax}\sin bx\,dx = \frac{a\sin bx - b\cos bx}{b^2}e^{ax} + C$$

$$\int e^{ax}\sin bx\,dx = \frac{a\sin bx - b\cos bx}{a^2+b^2}e^{ax} + C$$

19. $\underbrace{\int e^{ax}\cos bx\,dx}_{(a,b\neq 0)} = \left\{ \begin{array}{l} u=\cos bx,\ dv=e^{ax}dx \\ du = -b\sin bx\,dx,\ v=\frac{1}{a}e^{ax} \end{array} \right\} =$

$= \frac{1}{a}\cos bx\, e^{ax} + \frac{b}{a}\int e^{ax}\sin bx\,dx = \left\{ \begin{array}{l} u=\sin bx,\ dv=e^{ax}dx \\ du = b\cos bx\,dx,\ v=\frac{1}{a}e^{ax} \end{array} \right\} =$

$= \frac{1}{a}\cos bx\, e^{ax} + \frac{b}{a}\left(\frac{1}{a}\sin bx\cdot e^{ax} - \frac{b}{a}\int e^{ax}\cos bx\,dx\right)$

$= \frac{1}{a}e^{ax}\cos bx + \frac{b}{a^2}e^{ax}\sin bx - \frac{b^2}{a^2}\underline{\int e^{ax}\cos bx\,dx}$

$\underbrace{\left(1+\frac{b^2}{a^2}\right)}_{\frac{a^2+b^2}{a^2}}\int e^{ax}\cos bx\,dx = \frac{a\cos bx + b\sin bx}{a^2}e^{ax} + C$

$\int e^{ax}\cos bx\,dx = \frac{a\cos bx + b\sin bx}{a^2+b^2}e^{ax} + C$

20. $\underline{\int \cos(\ln x)\,dx} = \left\{ \begin{array}{l} u=\cos(\ln x),\ dv=dx \\ du = -\sin(\ln x)\cdot\frac{1}{x}dx,\ v=x \end{array} \right\} =$

$= x\cos(\ln x) + \int \sin(\ln x)\,dx = \left\{ \begin{array}{l} u=\sin(\ln x),\ dv=dx \\ du = \cos(\ln x)\cdot\frac{1}{x}dx,\ v=x \end{array} \right\} =$

$= x\cos(\ln x) + x\sin(\ln x) - \underline{\int\cos(\ln x)\,dx}$

$\int\cos(\ln x)\,dx = \frac{\cos(\ln x)+\sin(\ln x)}{2}x + C$

21. $\int x\sin\sqrt{x}\,dx = \left\{ \begin{array}{l} \sqrt{x}=t \\ \frac{dx}{2\sqrt{x}}=dt \end{array} \right\} = \int t^2\sin t\cdot 2t\,dt = 2\int t^3\sin t\,dt =$

$= \left\{ \begin{array}{l} u=t^3,\ dv=\sin t\,dt \\ du=3t^2dt,\ v=-\cos t \end{array} \right\} = -2t^3\cos t + 2\cdot 3\int t^2\cos t\,dt =$

$= \left\{ \begin{array}{l} u=t^2,\ dv=\cos t\,dt \\ du=2t\,dt,\ v=\sin t \end{array} \right\} = -2t^3\cos t + 6t^2\sin t - 12\int t\sin t\,dt =$

$= \left\{ \begin{array}{l} u=t,\ dv=\sin t\,dt \\ du=dt,\ v=-\cos t \end{array} \right\} = -2t^3\cos t + 6t^2\sin t + 12t\cos t - \int\cos t\,dt =$

$= -2t^3\cos t + 6t^2\sin t + 12t\cos t - \sin t + C =$

$= -2x\sqrt{x}\cos\sqrt{x} + 6x\sin\sqrt{x} + 12\sqrt{x}\cos\sqrt{x} - \sin\sqrt{x} + C.$