Интеграција рационалних ф-ја

- $P_n(x)$ – полином степена n

  $Q_m(x)$ – полином степена m

  Рационална функција је ф-ја облика $R(x) = \dfrac{P_n(x)}{Q_m(x)}$.

  Ако је $n < m$, онда кажемо да је $R(x)$ права рационална функција.

- Општи полином степена n:

  $n=0$: A (константа)

  $n=1$: $Ax+B$

  $n=2$: Ax^2+Bx+C

  $n=3$: Ax^3+Bx^2+Cx+D

  $\vdots$

- Растављање рационалне ф-је (опис најтеже ситуације)

$$R(x) = \frac{P_n(x)}{Q_m(x)} = \frac{P_n(x)}{(a_1x-b_1)^{r_1}\cdots(a_Nx-b_N)^{r_N}(c_1x^2+d_1x+e_1)^{s_1}\cdots(c_Mx^2+d_Mx+e_M)^{s_M}} =$$

$$= \frac{A_{1,1}}{a_1x-b_1} + \cdots + \frac{A_{1,r_1}}{(a_1x-b_1)^{r_1}} + \cdots + \frac{A_{N,1}}{a_Nx-b_N} + \cdots + \frac{A_{N,r_N}}{(a_Nx-b_N)^{r_N}} +$$

$$+ \frac{B_{1,1}x+C_{1,1}}{c_1x^2+d_1x+e_1} + \cdots + \frac{B_{1,s_1}x+C_{1,s_1}}{(c_1x^2+d_1x+e_1)^{s_1}} + \cdots$$

$$\cdots + \frac{B_{M,1}x+C_{M,1}}{c_Mx^2+d_Mx+e_M} + \cdots + \frac{B_{M,s_M}x+C_{M,s_M}}{(c_Mx^2+d_Mx+e_M)^{s_M}}$$

1. $I = \int \frac{x\,dx}{x^2-2x-3}$

$$\frac{x}{x^2-2x-3} = \frac{x}{(x-3)(x+1)} = \frac{A}{x-3} + \frac{B}{x+1} \quad / (x-3)(x+1)$$

$$x = A(x+1) + B(x-3)$$

$$\left.\begin{aligned} x:&\quad 1 = A+B \\ 1:&\quad 0 = A-3B \end{aligned}\right\}(-) \qquad 1 = 4B \Rightarrow B = 1/4,\ A = 3/4$$

$$I = \int\left(\frac{3/4}{x-3} + \frac{1/4}{x+1}\right)dx = \frac{3}{4}\ln|x-3| + \frac{1}{4}\ln|x+1| + C$$

2. $I = \int \frac{3-2x}{x^2-x-2}\,dx$

$$\frac{3-2x}{x^2-x-2} = \frac{3-2x}{(x-2)(x+1)} = \frac{A}{x-2} + \frac{B}{x+1} \quad / (x-2)(x+1)$$

$$3-2x = A(x+1) + B(x-2)$$

$$\left.\begin{aligned} x:&\quad -2 = A+B \\ 1:&\quad 3 = A-2B \end{aligned}\right\}(-) \qquad -5 = 3B \Rightarrow B = -5/3$$

$$A = -2 + \frac{5}{3} = -\frac{1}{3}$$

$$I = \int\left(\frac{-1/3}{x-2} + \frac{-5/3}{x+1}\right)dx = -\frac{1}{3}\ln|x-2| - \frac{5}{3}\ln|x+1| + C$$

3. $I = \int \frac{dx}{1-x^2}$

$$\frac{1}{1-x^2} = \frac{1}{(1-x)(1+x)} = \frac{A}{1-x} + \frac{B}{1+x} \quad / (1-x)(1+x)$$

$$1 = A(1+x) + B(1-x)$$

$$\left.\begin{aligned} x:&\quad 0 = A-B \\ 1:&\quad 1 = A+B \end{aligned}\right\}(+) \qquad 2A = 1 \Rightarrow A = 1/2,\ B = 1/2$$

$$I = \int\left(\frac{1/2}{1-x} + \frac{1/2}{1+x}\right)dx = -\frac{1}{2}\ln|x-1| + \frac{1}{2}\ln|x+1| + C = \frac{1}{2}\ln\left|\frac{x+1}{x-1}\right| + C$$

4. $I = \int \frac{dx}{x^2 - a^2}$

$$\frac{1}{x^2-a^2} = \frac{1}{(x-a)(x+a)} = \frac{A}{x-a} + \frac{B}{x+a} \quad / (x-a)(x+a)$$

$$1 = A(x+a) + B(x-a)$$

$$\left.\begin{array}{ll} x: & 0 = A + B \quad / \cdot a \\ 1: & 1 = aA - aB \end{array}\right\} (+) \qquad 2aA = 1 \Rightarrow A = 1/(2a), \; B = -1/(2a)$$

$$I = \int \left(\frac{1/(2a)}{x-a} + \frac{-1/(2a)}{x+a} \right) dx = \frac{1}{2a} \ln|x-a| - \frac{1}{2a} \ln|x+a| + C$$

$$= \frac{1}{2a} \ln\left|\frac{x-a}{x+a}\right| + C$$

5. $I = \int \frac{3x^2 + 2x - 3}{x^3 - x} dx$

$$\frac{3x^2+2x-3}{x^3-x} = \frac{3x^2+2x-3}{x(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x} + \frac{C}{x+1} \quad / (x-1)x(x+1)$$

$$\underline{3x^2} + \underline{\underline{2x}} - 3 = Ax(x+1) + B(x-1)(x+1) + Cx(x-1)$$

$$= \underline{Ax^2} + \underline{\underline{Ax}} + \underline{Bx^2} - B + \underline{Cx^2} - \underline{\underline{Cx}}$$

$$\left.\begin{array}{ll} x^2: & 3 = A + B + C \\ x: & 2 = A - C \end{array}\right\} \Rightarrow \left.\begin{array}{l} A + C = 0 \\ A - C = 2 \end{array}\right\} (+) \qquad \begin{array}{l} 2A = 2 \Rightarrow A = 1 \\ C = -1 \end{array}$$

$$1: \quad -3 = -B \Rightarrow B = 3$$

$$I = \int \left(\frac{1}{x-1} + \frac{3}{x} + \frac{-1}{x+1} \right) dx = \ln|x-1| + 3\ln|x| - \ln|x+1| + C$$

$$= \ln\left|\frac{x^3(x-1)}{x+1}\right| + C$$

6. $I = \int \frac{2x^2 dx}{x^4-1}$

$$\frac{2x^2}{x^4-1} = \frac{2x^2}{(x-1)(x+1)(x^2+1)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{Cx+D}{x^2+1} \quad /(x^4-1)$$

$$2x^2 = A\underbrace{(x+1)(x^2+1)}_{x^3+x^2+x+1} + B\underbrace{(x-1)(x^2+1)}_{x^3-x^2+x-1} + (Cx+D)\underbrace{(x-1)(x+1)}_{x^2-1}$$

$$= Ax^3 + Ax^2 + Ax + \boxed{A} + Bx^3 - Bx^2 + Bx \boxed{-B} + Cx^3 - Cx + Dx^2 \boxed{-D}$$

$x^3: \quad 0 = A + B + C \quad (+) \quad 0 = 2A + 2B$

$x^2: \quad 2 = A - B + D \quad (+) \quad 2 = 2A - 2B \quad \Big\}(+) \quad 2 = 4A \Rightarrow A = 1/2$

$x: \quad 0 = A + B - C \qquad B = -1/2$

$1: \quad 0 = A - B - D \qquad C = 0$

$D = 1$

$$I = \int\left(\frac{1/2}{x-1} + \frac{-1/2}{x+1} + \frac{1}{x^2+1}\right)dx = \frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| + \operatorname{arctg} x + C =$$

$$= \frac{1}{2}\ln\left|\frac{x-1}{x+1}\right| + \operatorname{arctg} x + C$$

7. $I = \int \frac{1-x}{x^2+2x+1}dx$

$$\frac{1-x}{x^2+2x+1} = \frac{1-x}{(x+1)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2} \quad /(x+1)^2$$

$$1-x = A(x+1) + B$$

$x: \quad -1 = A$

$1: \quad 1 = A + B \Rightarrow B = 2$

$$I = \int\left(\frac{-1}{x+1} + \frac{2}{(x+1)^2}\right)dx = -\ln|x+1| + 2\frac{(x+1)^{-1}}{-1} + C$$

$$= -\ln|x+1| - \frac{2}{x+1} + C$$

8. $I = \int \frac{dx}{x^2(x-1)}$

$$\frac{1}{x^2(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} \quad / x^2(x-1)$$

$$1 = A\underbrace{x(x-1)}_{x^2-x} + B(x-1) + Cx^2$$

x^2: $0 = A + C \Rightarrow C = 1$

x: $0 = -A - B \Rightarrow A = -1$

1: $1 = -B \Rightarrow B = 1$

$$I = \int\left(\frac{-1}{x} + \frac{1}{x^2} + \frac{1}{x-1}\right)dx = -\ln|x| - \frac{1}{x} + \ln|x-1| + C$$

$$= \ln\left|\frac{x-1}{x}\right| - \frac{1}{x} + C$$

9. $I = \int \frac{dx}{x(x^2+1)^2}$

$$\frac{1}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2} \quad / x(x^2+1)^2$$

$$1 = A\underbrace{(x^2+1)^2}_{x^4+2x^2+1} + (Bx+C)\underbrace{x(x^2+1)}_{x^3+x} + (Dx+E)x$$

$$= Ax^4 + 2Ax^2 + A + Bx^4 + Bx^2 + Cx^3 \boxed{+Cx} + Dx^2 \boxed{+Ex}$$

x^4: $0 = A + B \Rightarrow B = -1$

x^3: $0 = C$

x^2: $0 = 2A + B + D \Rightarrow D = -1$

x: $0 = C + E \Rightarrow E = 0$

1: $1 = A$

$$I=\int\left(\frac{1}{x}+\frac{-x}{x^2+1}+\frac{-x}{(x^2+1)^2}\right)dx=\left\{\begin{matrix}x^2+1=t\\ 2x\,dx=dt\end{matrix}\right\}=$$

$$=\int\frac{dx}{x}-\int\frac{dt/2}{t}-\int\frac{dt/2}{t^2}=\ln|x|-\frac{1}{2}\ln|t|-\frac{1}{2}\left(-\frac{1}{t}\right)+C$$

$$=\ln|x|-\frac{1}{2}\ln(x^2+1)+\frac{1}{2(x^2+1)}+C$$

10. $$I=\int\frac{x^3+1}{x^2-3x+2}\,dx$$

$$(x^3+1):(x^2-3x+2)=x+3+\frac{7x-5}{x^2-3x+2}$$

$$\begin{array}{l}\underline{x^3-3x^2+2x}\\ /\ \ 3x^2+2x+1\\ \ \ \ \underline{3x^2-9x+6}\\ \ \ /\ \ \ \ \ 7x-5\end{array}$$

$$I=\int\left(x+3+\frac{7x-5}{(x-2)(x-1)}\right)dx$$

$$\frac{7x-5}{(x-2)(x-1)}=\frac{A}{x-2}+\frac{B}{x-1}\quad/(x-2)(x-1)$$

$$7x-5=A(x-1)+B(x-2)$$

$$\left.\begin{matrix}x: & 7=A+B\\ 1: & -5=-A-2B\end{matrix}\right\}(+)\qquad 2=-B\Rightarrow B=-2,\ A=9$$

$$I=\int\left(x+3+\frac{9}{x-2}+\frac{-2}{x-1}\right)dx=\frac{x^2}{2}+3x+9\ln|x-2|-2\ln|x-1|+C$$

$\boxed{11}$ $$\int\underbrace{\frac{dx}{1+2x^2}}_{\text{нерастовљив}}=\int\frac{dx}{1+(x\sqrt{2})^2}=\left\{\begin{matrix}x\sqrt{2}=t\\ \sqrt{2}dx=dt\end{matrix}\right\}=\int\frac{dt/\sqrt{2}}{1+t^2}=$$

$$=\frac{1}{\sqrt{2}}\operatorname{arctg}t+C=\frac{1}{\sqrt{2}}\operatorname{arctg}(x\sqrt{2})+C$$

12. $I = \int \frac{dx}{\underbrace{x^2-x+1}_{\text{нерастављив}}}$

$$x^2-x+1 = x^2 - 2\cdot\frac{1}{2}x + \frac{1}{4} - \frac{1}{4} + 1 = \left(x-\frac{1}{2}\right)^2 + \frac{3}{4} =$$

$$= \frac{3}{4}\left(\frac{4}{3}\left(x-\frac{1}{2}\right)^2+1\right) = \frac{3}{4}\left(\left(\frac{\cancel{2}}{\sqrt{3}}\cdot\frac{2x-1}{\cancel{2}}\right)^2+1\right)$$

$$I = \int \frac{dx}{\frac{3}{4}\left(\left(\frac{2x-1}{\sqrt{3}}\right)^2+1\right)} = \left\{\begin{array}{l} \frac{2x-1}{\sqrt{3}} = t \\ \frac{2dx}{\sqrt{3}} = dt \end{array}\right\} = \frac{4}{3}\int \frac{\sqrt{3}/2\, dt}{t^2+1} =$$

$$= \frac{2}{\sqrt{3}}\operatorname{arctg} t + C = \frac{2}{\sqrt{3}}\operatorname{arctg}\frac{2x-1}{\sqrt{3}} + C$$

13. $I = \int \frac{dx}{2x^2+3x+2}$

$2x^2+3x+2 = 0 \Rightarrow x_{1,2} = \frac{-3\pm\sqrt{9-16}}{2} \notin \mathbb{R}$; нерастављив

$$2x^2+3x+2 = 2\left(x^2+\frac{3}{2}x+1\right) = 2\left(x^2+2\cdot\frac{3}{4}x+\frac{9}{16}-\frac{9}{16}+1\right) =$$

$$= 2\left(\left(x+\frac{3}{4}\right)^2+\frac{7}{16}\right) = 2\left(\frac{7}{16}\left(\frac{16}{7}\left(\frac{4x+3}{4}\right)^2+1\right)\right) =$$

$$= \cancel{2}\,\frac{7}{8}\left(\left(\frac{\cancel{4}}{\sqrt{7}}\cdot\frac{4x+3}{\cancel{4}}\right)^2+1\right)$$

$$I = \int \frac{dx}{\frac{7}{8}\left(\left(\frac{4x+3}{\sqrt{7}}\right)^2+1\right)} = \left\{\begin{array}{l} \frac{4x+3}{\sqrt{7}} = t \\ \frac{4dx}{\sqrt{7}} = dt \end{array}\right\} =$$

$$= \frac{8}{7}\int \frac{\sqrt{7}/4\, dt}{t^2+1} = \frac{2}{\sqrt{7}}\operatorname{arctg} t + C = \frac{2}{\sqrt{7}}\operatorname{arctg}\frac{4x+3}{\sqrt{7}} + C$$

14. $I = \int \frac{dx}{3+4x-4x^2}$

$3+4x-4x^2 = 0 \Rightarrow x_{1,2} = \frac{-4\pm\sqrt{16-48}}{-8} \notin \mathbb{R}$; нерастављив.

14. $I = \int \frac{x^4+1}{x^3-1}\,dx$

$$(x^4+1):(x^3-1) = x + \frac{x+1}{x^3-1}$$
$$\frac{x^4 - x}{\quad x+1}$$

$$I = \int \left(x + \frac{x+1}{x^3-1} \right) dx$$

$$\frac{x+1}{x^3-1} = \frac{x+1}{(x-1)(x^2+x+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1} \quad \Big/ (x-1)(x^2+x+1)$$

$$x+1 = A(x^2+x+1) + (Bx+C)(x-1) = Ax^2 + Ax + A + Bx^2 - Bx + Cx - C$$

$$\begin{array}{lll} x^2: & 0 = A + B \Rightarrow B = -A & \\ x: & 1 = A - B + C & \left.\begin{array}{l} 2A + C = 1 \\ A - C = 1 \end{array}\right\} (+) \quad 3A = 2 \Rightarrow A = 2/3 \\ 1: & 1 = A - C & \qquad B = -2/3 \\ & & \qquad C = -1/3 \end{array}$$

$$I = \int \left(x + \frac{2/3}{x-1} + \frac{-2/3\,x - 1/3}{x^2+x+1} \right) dx =$$
$$= \int \left(x + \frac{2}{3} \cdot \frac{1}{x-1} - \frac{1}{3} \frac{2x+1}{x^2+x+1} \right) dx = \left| \begin{array}{l} x^2+x+1 = t \\ (2x+1)\,dx = dt \end{array} \right| =$$
$$= \int \left(x + \frac{1}{3} \frac{1}{x-1} \right) dx - \frac{1}{3} \int \frac{dt}{t} = \frac{x^2}{2} + \frac{1}{3} \ln|x-1| - \frac{1}{3} \ln|t| + C$$
$$= \frac{x^2}{2} + \frac{1}{3} \ln \frac{|x-1|}{x^2+x+1} + C$$

15. $I = \int \frac{dx}{(x^2+1)^2}$

$$\frac{1}{(x^2+1)^2} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2} \quad \Big/ (x^2+1)^2$$
$$1 = (Ax+B)(x^2+1) + Cx + D$$
$$= Ax^3 + Ax + Bx^2 + B + Cx + D$$

$x^3: \quad 0 = A$

$x^2: \quad 0 = B$

$x: \quad 0 = A + C \Rightarrow C = 0$

$1: \quad 1 = B + D \Rightarrow D = 1$

Враћамо се на почетак,

$$I = \int \frac{dx}{(x^2+1)^2}$$

$$I=\int\frac{(x^2+1)-x^2}{(x^2+1)^2}\,dx=\int\frac{dx}{x^2+1}-\int x\cdot\frac{x\,dx}{(x^2+1)^2}=$$

$$=\left\{\begin{array}{ll} u=x, & dv=\frac{x\,dx}{(x^2+1)^2} \\ du=dx, & v=\int\frac{x\,dx}{(x^2+1)^2}=\left\{\begin{array}{l} x^2+1=t \\ 2x\,dx=dt\end{array}\right\}=\int\frac{dt/2}{t^2}=\frac{1}{2}\left(-\frac{1}{t}\right)=-\frac{1}{2(x^2+1)}\end{array}\right\}=$$

$$=\int\frac{dx}{x^2+1}+\frac{x}{2(x^2+1)}-\frac{1}{2}\int\frac{dx}{x^2+1}=\frac{x}{2(x^2+1)}+\frac{1}{2}\int\frac{dx}{x^2+1}=$$

$$=\frac{1}{2}\left(\frac{x}{x^2+1}+\operatorname{arctg}x\right)+C$$

16. $I=\int\frac{dx}{(2x^2+x+1)^3}$

$$2x^2+x+1=2\left(x^2+\frac{1}{2}x+\frac{1}{2}\right)=2\left(x^2+2\cdot\frac{1}{4}x+\frac{1}{16}-\frac{1}{16}+\frac{1}{2}\right)=$$

$$=2\left(\left(x+\frac{1}{4}\right)^2+\frac{7}{16}\right)=2\cdot\left(\frac{7}{16}\left(\frac{16}{7}\left(x+\frac{1}{4}\right)^2+1\right)\right)=$$

$$=\frac{7}{8}\left(\left(\frac{\cancel{4}}{\sqrt{7}}\,\frac{4x+1}{\cancel{4}}\right)^2+1\right)$$

$$I=\int\frac{dx}{\left(\frac{7}{8}\left(\left(\frac{4x+1}{\sqrt{7}}\right)^2+1\right)\right)^3}=\left\{\begin{array}{l}\frac{4x+1}{\sqrt{7}}=t \\ \frac{4dx}{\sqrt{7}}=dt\end{array}\right\}=\frac{8^3}{7^3}\int\frac{\frac{\sqrt{7}}{4}dt}{(t^2+1)^3}=$$

$$=\frac{2^7}{\sqrt{7}^5}\int\frac{(t^2+1)-t^2}{(t^2+1)^3}\,dt=\frac{2^7}{\sqrt{7}^5}\left(\int\frac{dt}{(t^2+1)^2}-\int t\cdot\frac{t\,dt}{(t^2+1)^3}\right)=$$

$$\left\{\begin{array}{ll} u=t, & dv=\frac{t\,dt}{(t^2+1)^3} \\ du=dt, & v=\int\frac{t\,dt}{(t^2+1)^3}=\left\{\begin{array}{l}t^2+1=s \\ 2t\,dt=ds\end{array}\right\}=\int\frac{ds/2}{s^3}=\frac{1}{2}\,\frac{s^{-2}}{-2}=-\frac{1}{4(t^2+1)^2}\end{array}\right\}=$$

$$=\frac{2^7}{\sqrt{7}^5}\left(\int\frac{dt}{(t^2+1)^2}+\frac{t}{4(t^2+1)^2}-\frac{1}{4}\int\frac{dt}{(t^2+1)^2}\right)=$$

$$=\frac{2^5}{\cancel{2}\sqrt{7}^5}\left(3\int\frac{dt}{(t^2+1)^2}+\frac{t}{\cancel{4}(t^2+1)^2}\right)=\boxed{15}=\left(\frac{2}{\cancel{2}\sqrt{7}}\right)^5\left(\frac{3}{2}\left(\frac{t}{t^2+1}+\operatorname{arctg}t\right)+\frac{t}{(t^2+1)^2}\right)+C=$$

$$=\left(\frac{2}{\sqrt{7}}\right)^5\cancel{3}\left(\frac{t}{(t^2+1)^2}+\frac{3t}{(t^2+1)\cdot 2}+\frac{3\operatorname{arctg}t}{2}\right)+C=$$

$$=\left(\frac{2}{\sqrt{17}}\right)^5\left(\frac{\frac{4x+1}{\sqrt{17}}}{\left(\left(\frac{4x+1}{\sqrt{17}}\right)^2+1\right)^2}+\frac{3\,\frac{4x+1}{\sqrt{17}}}{2\left(\left(\frac{4x+1}{\sqrt{17}}\right)^2+1\right)}+\frac{3\,\mathrm{arctg}\,\frac{4x+1}{\sqrt{17}}}{2}\right)+C$$

(Видети и $\boxed{3}$ рек. ф-ле)

17. $$I=\int\frac{x^2-1}{x^4+1}\,dx=\int\frac{x^2\left(1-\frac{1}{x^2}\right)}{x^2\left(x^2+\frac{1}{x^2}\right)}\,dx=\int\frac{1-\frac{1}{x^2}}{\left(x+\frac{1}{x}\right)^2-2}\,dx=$$

$$=\left\{\begin{array}{l} x+\frac{1}{x}=t \\ \left(1-\frac{1}{x^2}\right)dx=dt\end{array}\right\}=\int\frac{dt}{t^2-2}$$

$$\frac{1}{t^2-2}=\frac{1}{(t-\sqrt2)(t+\sqrt2)}=\frac{A}{t-\sqrt2}+\frac{B}{t+\sqrt2}\quad /(t-\sqrt2)(t+\sqrt2)$$

$$1=A(t+\sqrt2)+B(t-\sqrt2)$$

$$\left.\begin{array}{ll} t: & 0=A+B \quad /\cdot\sqrt2 \\ 1: & 1=A\sqrt2-B\sqrt2 \end{array}\right\}(+) \qquad \begin{array}{l} 2A\sqrt2=1 \Rightarrow A=1/(2\sqrt2) \\ B=-1/(2\sqrt2)\end{array}$$

$$I=\int\left(\frac{1/(2\sqrt2)}{t-\sqrt2}+\frac{-1/(2\sqrt2)}{t+\sqrt2}\right)dt=\frac{1}{2\sqrt2}\ln|t-\sqrt2|-\frac{1}{2\sqrt2}\ln|t+\sqrt2|+C$$

$$=\frac{1}{2\sqrt2}\ln\left|\frac{t-\sqrt2}{t+\sqrt2}\right|+C=\frac{1}{2\sqrt2}\ln\left|\frac{x+\frac{1}{x}-\sqrt2}{x+\frac{1}{x}+\sqrt2}\right|+C$$

18. $$\int\frac{x^2+1}{x^4+1}\,dx=\int\frac{x^2\left(1+\frac{1}{x^2}\right)}{x^2\left(x^2+\frac{1}{x^2}\right)}\,dx=\int\frac{1+\frac{1}{x^2}}{\left(x-\frac{1}{x}\right)^2+2}\,dx=$$

$$=\left\{\begin{array}{l} x-\frac{1}{x}=t \\ \left(1+\frac{1}{x^2}\right)dx=dt\end{array}\right\}=\int\frac{dt}{t^2+2}=\int\frac{dt}{2\left(\left(\frac{t}{\sqrt2}\right)^2+1\right)}=\left\{\begin{array}{l} \frac{t}{\sqrt2}=s \\ dt/\sqrt2=ds\end{array}\right\}=$$

$$=\int\frac{\sqrt2\,ds}{2(s^2+1)}=\frac{1}{\sqrt2}\,\mathrm{arctg}\,s+C=\frac{1}{\sqrt2}\,\mathrm{arctg}\,\frac{t}{\sqrt2}+C=\frac{1}{\sqrt2}\,\mathrm{arctg}\,\frac{x-\frac{1}{x}}{\sqrt2}+C$$

19. $\int \frac{dx}{x^4+1} = \int \frac{\frac{1}{2}((x^2+1)-(x^2-1))}{x^4+1} dx = \frac{1}{2}\Big(\underbrace{\int \frac{x^2+1}{x^4+1} dx}_{\boxed{18.}} - \underbrace{\int \frac{x^2-1}{x^4+1} dx}_{\boxed{17.}}\Big) = \dots$

$\dots = \frac{1}{4\sqrt{2}} \ln \left| \frac{x+\frac{1}{x}-\sqrt{2}}{x+\frac{1}{x}+\sqrt{2}} \right| - \frac{1}{2\sqrt{2}} \operatorname{arctg} \frac{x-\frac{1}{x}}{\sqrt{2}} + C$

20. $I = \int \frac{e^x dx}{2e^{2x}-7e^x+3} = \left\{ \begin{array}{l} e^x = t \\ e^x dx = dt \end{array} \right\} = \int \frac{dt}{2t^2-7t+3}$

$2t^2-7t+3=0 \Rightarrow t_{1/2} = \frac{7 \pm \sqrt{49-24}}{4} = \frac{7\pm 5}{4} \begin{array}{l} \nearrow 3 \\ \searrow 1/2 \end{array}$

$2t^2-7t+3 = 2(t-3)(t-\frac{1}{2}) = (2t-1)(t-3)$

$\frac{1}{2t^2-7t+3} = \frac{1}{(2t-1)(t-3)} = \frac{A}{2t-1} + \frac{B}{t-3} \quad / (2t-1)(t-3)$

$1 = A(t-3) + B(2t-1)$

$t: \quad 0 = A + 2B$

$1: \quad 1 = -3A - B \quad / \cdot 2 \quad (+)$

$2 = -5A \Rightarrow A = -2/5$

$B = 1/5$

$I = \int \left(\frac{-2/5}{2t-1} + \frac{1/5}{t-3} \right) dt = -\frac{2}{5} \cdot \frac{1}{2} \ln|2t-1| + \frac{1}{5} \ln|t-3| + C =$

$= \frac{1}{5} \ln \left| \frac{t-3}{2t-1} \right| + C = \frac{1}{5} \ln \left| \frac{e^x-3}{2e^x-1} \right| + C$

21. $I = \int \frac{2\sin x}{\cos^4 x - 3\cos^2 x - 4} dx = \left\{ \begin{array}{l} \cos x = t \\ -\sin x dx = dt \end{array} \right\} = \int \frac{2(-dt)}{t^4-3t^2-4}$

$\frac{1}{t^4-3t^2-4} = \frac{1}{(t^2-4)(t^2+1)} = \frac{1}{(t-2)(t+2)(t^2+1)} =$

$= \frac{A}{t-2} + \frac{B}{t+2} + \frac{Ct+D}{t^2+1} \quad / (t-2)(t+2)(t^2+1)$

$$\boxed{1} = A\underbrace{(t+2)(t^2+1)}_{t^3+2t^2+t+2} + B\underbrace{(t-2)(t^2+1)}_{t^3-2t^2+t-2} + (Ct+D)\underbrace{(t-2)(t+2)}_{t^2-4}$$

$$= \underline{At^3} + \underline{\underline{2At^2}} + At + \boxed{2A} + \underline{Bt^3} - \underline{\underline{2Bt^2}} + Bt \boxed{-2B} + \underline{Ct^3} - 4Ct + \underline{\underline{Dt^2}} \boxed{-4D}$$

t^3: $0 = A + B + C$ $\quad |\cdot 4 \quad -(+)$ $\quad 5A + 5B = 0 \quad |\cdot 2$ $\quad (+)$ $\quad 20A = 1$

t^2: $0 = 2A - 2B + D$ $\quad |\cdot 4 \quad -(+)$ $\quad 10A - 10B = 1$ $\quad A = 1/20$

t: $0 = A + B - 4C$ $\quad B = -1/20$

1: $1 = 2A - 2B - 4D$ $\quad C = 0$

$D = -1/5$

$$I = \int\left(\frac{1/20}{t-2} + \frac{-1/20}{t+2} + \frac{-1/5}{t^2+1}\right)dt = \frac{1}{20}\ln|t-2| - \frac{1}{20}\ln|t+2|$$

$$- \frac{1}{5}\operatorname{arctg} t + C =$$

$$= \frac{1}{20}\ln\left|\frac{t-2}{t+2}\right| - \frac{1}{5}\operatorname{arctg} t + C =$$

$$= \frac{1}{20}\ln\left|\frac{\cos x - 2}{\cos x + 2}\right| - \frac{1}{5}\operatorname{arctg}(\cos x) + C$$