Интеграција тригонометријских функција

- $$\int R(\sin x, \cos x)\,dx = \left\{ \begin{array}{l} \boxed{tg\frac{x}{2}=t},\quad dx=\frac{2dt}{1+t^2} \\ \sin x = \frac{2t}{1+t^2},\quad \cos x = \frac{1-t^2}{1+t^2} \end{array} \right\} =$$

$$= \int R\left(\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2}\right)\frac{2dt}{1+t^2} = \int R^*(t)\,dt$$

- $$\int R(\sin^2 x, \cos^2 x)\,dx = \left\{ \begin{array}{l} \boxed{tg\, x=t},\quad dx=\frac{dt}{1+t^2} \\ \sin^2 x = \frac{t^2}{1+t^2},\quad \cos^2 x = \frac{1}{1+t^2} \end{array} \right\} =$$

$$= \int R\left(\frac{t^2}{1+t^2}, \frac{1}{1+t^2}\right)\frac{dt}{1+t^2} = \int R^*(t)\,dt$$

- $$\int R(\sin x)\cos x\,dx = \left\{ \begin{array}{l} \sin x = t \\ \cos x\,dx = dt \end{array} \right\} = \int R(t)\,dt$$

- $$\int R(\cos x)\sin x\,dx = \left\{ \begin{array}{l} \cos x = t \\ -\sin x\,dx = dt \end{array} \right\} = \int R(t)(-dt)$$

1. $$\int \frac{dx}{\sin x} = \left\{ \begin{array}{l} tg\frac{x}{2}=t,\quad dx=\frac{2dt}{1+t^2} \\ \sin x = \frac{2t}{1+t^2} \end{array} \right\} = \int \frac{\frac{\cancel{2}dt}{\cancel{1+t^2}}}{\frac{\cancel{2}t}{\cancel{1+t^2}}} = \ln|t| + C =$$

$$= \ln\left|tg\frac{x}{2}\right| + C$$

2. $$\int \frac{dx}{2\sin x - \cos x + 3} = \left\{ \begin{array}{l} tg\frac{x}{2}=t,\quad dx=\frac{2dt}{1+t^2} \\ \sin x = \frac{2t}{1+t^2},\quad \cos x = \frac{1-t^2}{1+t^2} \end{array} \right\} =$$

$$= \int \frac{\frac{2dt}{1+t^2}}{2\frac{2t}{1+t^2} - \frac{1-t^2}{1+t^2} + 3} = \int \frac{\frac{2dt}{\cancel{1+t^2}}}{\frac{4t-1+t^2+3+3t^2}{\cancel{1+t^2}}} = \int \frac{2dt}{4t^2+4t+2} =$$

$$= \int \frac{dt}{2t^2+2t+1} =$$

$2t^2+2t+1=0 \Rightarrow t_{1,2}=\frac{-2\pm\sqrt{4-8}}{4}\notin \mathbb{R}$ (нерастовимо)

$2t^2+2t+1=2(t^2+t+\frac{1}{2})=2(t^2+2\cdot\frac{1}{2}t+\frac{1}{4}-\frac{1}{4}+\frac{1}{2})=$

$=2((t+\frac{1}{2})^2+\frac{1}{4})=2(\frac{1}{4}(4(t+\frac{1}{2})^2+1)=\frac{1}{2}((\cancel{2}\cdot\frac{2t+1}{\cancel{2}})^2+1)$

$\stackrel{=}{}\int\frac{dt}{\frac{1}{2}((2t+1)^2+1)}=\left|\begin{matrix}2t+1=s\\2dt=ds\end{matrix}\right|=\int\frac{ds}{s^2+1}=\operatorname{arctg}s+C=$

$=\operatorname{arctg}(2t+1)+C=\operatorname{arctg}(2\operatorname{tg}\frac{x}{2}+1)+C.$

3. $\int\frac{dx}{(2+\cos x)\sin x}=\left|\begin{matrix}\operatorname{tg}\frac{x}{2}=t, & dx=\frac{2dt}{1+t^2}\\ \sin x=\frac{2t}{1+t^2}, & \cos x=\frac{1-t^2}{1+t^2}\end{matrix}\right|=$

$=\int\frac{\cancel{2}\frac{dt}{\cancel{1+t^2}}}{(2+\frac{1-t^2}{1+t^2})\frac{\cancel{2}t}{\cancel{1+t^2}}}=\int\frac{dt}{\frac{2+2t^2+1-t^2}{1+t^2}t}=\int\frac{(1+t^2)dt}{t(t^2+3)}\stackrel{=}{}$

$\frac{t^2+1}{t(t^2+3)}=\frac{A}{t}+\frac{Bt+C}{t^2+3}\quad/t(t^2+3)$

$t^2+1=A(t^2+3)+Bt^2+Ct$

$t^2:\quad 1=A+B\Rightarrow B=\frac{2}{3}$

$t:\quad 0=C$

$1:\quad 1=3A\Rightarrow A=\frac{1}{3}$

$\stackrel{=}{}\int\left(\frac{1/3}{t}+\frac{2/3\ t}{t^2+3}\right)dt=\left|\begin{matrix}t^2+3=s\\2t\,dt=ds\end{matrix}\right|=\frac{1}{3}\int\frac{dt}{t}+\frac{1}{3}\int\frac{ds}{s}=$

$=\frac{1}{3}\ln|t|+\frac{1}{3}\ln\underbrace{|s|}_{+C}=\frac{1}{3}\ln\underbrace{|t(t^2+3)|}_{+C}=\frac{1}{3}\ln\left|\operatorname{tg}\frac{x}{2}(\operatorname{tg}^2\frac{x}{2}+3)\right|+C$

4. $$\int\frac{dx}{\sin^2x+2\cos^2x}=\left\{\begin{matrix} tg\,x=t, & dx=\frac{dt}{1+t^2}\\ \sin^2x=\frac{t^2}{1+t^2}, & \cos^2x=\frac{1}{1+t^2}\end{matrix}\right\}=$$

$$=\int\frac{\frac{dt}{\cancel{1+t^2}}}{\frac{t^2}{\cancel{1+t^2}}+2\cdot\frac{1}{\cancel{1+t^2}}}=\int\frac{dt}{t^2+2}=\int\frac{dt}{2\left(\left(\frac{t}{\sqrt2}\right)^2+1\right)}=\left\{\begin{matrix}\frac{t}{\sqrt2}=s\\ \frac{dt}{\sqrt2}=ds\end{matrix}\right\}=$$

$$=\int\frac{\sqrt2\,ds}{2(s^2+1)}=\frac{1}{\sqrt2}\,arctg\,s+C=\frac{1}{\sqrt2}\,arctg\,\frac{t}{\sqrt2}+C=$$

$$=\frac{1}{\sqrt2}\,arctg\,\frac{tg\,x}{\sqrt2}+C$$

(6) 5. $$\int\frac{dx}{\sin^4x\cos^2x}=\left\{\begin{matrix} tg\,x=t, & dx=\frac{dt}{1+t^2}\\ \sin^2x=\frac{t^2}{1+t^2}, & \cos^2x=\frac{1}{1+t^2}\end{matrix}\right\}=$$

$$=\int\frac{\frac{dt}{\cancel{1+t^2}}}{\frac{t^4}{(1+t^2)^2}\cdot\frac{1}{\cancel{1+t^2}}}=\int\frac{t^4+2t^2+1}{t^4}\,dt=\int(1+2t^{-2}+t^{-4})\,dt=$$

$$\cancel{=t+\frac{2}{3}\,\frac{1}{t^3}}\quad=t+\frac{2}{t}-\frac{1}{3t^3}+C=$$

$$=tg\,x-\frac{2}{tg\,x}-\frac{1}{3tg^3x}+C$$

(5) 6. $$\int\frac{dx}{\cos^4x}=\left\{\begin{matrix} tg\,x=t, & dx=\frac{dt}{1+t^2}\\ \cos^2x=\frac{1}{1+t^2} & \end{matrix}\right\}=\int\frac{\frac{dt}{\cancel{1+t^2}}}{\frac{1}{\cancel{(1+t^2)^2}}}=$$

$$=\int(t^2+1)\,dt=\frac{t^3}{3}+t+C=\frac{tg^3x}{3}+tg\,x+C$$

7. $$\int\frac{1+\cos^2x}{\cos^2x}\sin x\,dx=\left\{\begin{matrix}\cos x=t\\ -\sin x\,dx=dt\end{matrix}\right\}=\int\frac{1+t^2}{t^2}(-dt)=$$

$$=-\int(t^{-2}+1)\,dt=\frac{1}{t}-t+C=\frac{1}{\cos x}-\cos x+C$$

8. $\int \frac{\cos^3 x}{\sin^2 x} dx = \int \frac{(1-\sin^2 x)\cos x}{\sin^2 x} dx = \begin{Bmatrix} \sin x = t \\ \cos x \, dx = dt \end{Bmatrix} =$

$= \int \frac{(1-t^2)\,dt}{t^2} = \int (t^{-2} - 1)\,dt = -\frac{1}{t} - t + C = -\frac{1}{\sin x} - \sin x + C$

9. $\int \sin^2 x \cos^3 x \, dx = \int \sin^2 x\,(1-\sin^2 x) \cos x \, dx = \begin{Bmatrix} \sin x = t \\ \cos x \, dx = dt \end{Bmatrix} =$

$= \int t^2(1-t^2)\,dt = \int (t^2 - t^4)\,dt = \frac{t^3}{3} - \frac{t^5}{5} + C =$

$= \frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + C$

10. $\int \frac{\sin 2x}{\cos^3 x} dx = \int \frac{2 \sin x \cos x}{\cos^3 x} dx = 2\int \frac{\sin x \, dx}{\cos^2 x} = \begin{Bmatrix} \cos x = t \\ -\sin x \, dx = dt \end{Bmatrix} =$

$= 2 \int \frac{-dt}{t^2} = 2\,\frac{1}{t} + C = \frac{2}{\cos t} + C$