Интеграција ирационалних функција

(I) $I = \int R\left(x, \sqrt[m]{\frac{ax+b}{cx+d}}\right) dx, \quad \neg(a=0 \wedge c=0)$

1) $ad \neq bc$

смена: $\frac{ax+b}{cx+d} = t^m \Rightarrow x = \dots, \; dx = \dots$

$I = \int R^*(t)\,dt$

2) $ad = bc \Rightarrow \frac{a}{c} = \frac{b}{d}$

$$\frac{ax+b}{cx+d} = \frac{\frac{ax+b}{cd}}{\frac{cx+d}{cd}} = \frac{\frac{1}{d}\left(\frac{a}{c}\right)x + \left(\frac{b}{d}\right)\frac{1}{c}}{\frac{1}{d}x + \frac{1}{c}} = \frac{\frac{a}{c}\cancel{\left(\frac{1}{d}x + \frac{1}{c}\right)}}{\cancel{\frac{1}{d}x + \frac{1}{c}}} = \frac{a}{c}$$

$I = \int R\left(x, \sqrt[m]{\frac{a}{c}}\right) dx = \int R^*(x)\,dx$

(2) 1. $I = \int \frac{1}{x}\sqrt{\frac{1-x}{1+x}}\,dx$

$a=-1, \; b=1, \; c=1, \; d=1; \quad ad = -1 \neq 1 = bc$

$m=2$

$\frac{1-x}{1+x} = t^2; \quad 1-x = t^2 + t^2x; \quad (t^2+1)x = 1-t^2$

$x = \frac{1-t^2}{1+t^2}$

$$dx = \frac{-2t(1+t^2)-(1-t^2)\cdot 2t}{(1+t^2)^2}\,dt = \frac{-2t\cancel{-2t^3}-2t\cancel{+2t^3}}{(1+t^2)^2}\,dt = \frac{-4t\,dt}{(1+t^2)^2}$$

$$I = \int \frac{1}{\frac{1-t^2}{\cancel{1+t^2}}}\sqrt{t^2}\,\frac{-4t\,dt}{(1+t^2)^{\cancel{2}}} = -4\int \frac{t^2\,dt}{(1-t)(1+t)(1+t^2)}$$

$$= 4\int \frac{t^2\,dt}{(t-1)(t+1)(t^2+1)}$$

$$\frac{t^2}{(t-1)(t+1)(t^2+1)} = \frac{A}{t-1} + \frac{B}{t+1} + \frac{Ct+D}{t^2+1} \quad \Big/ (t-1)(t+1)(t^2+1)$$

$$t^2 = A\underbrace{(t+1)(t^2+1)}_{t^3+t^2+t+1} + B\underbrace{(t-1)(t^2+1)}_{t^3-t^2+t-1} + (Ct+D)\underbrace{(t-1)(t+1)}_{t^2-1}$$

$$t^2 = At^3 + At^2 + At + \boxed{A} + Bt^3 - Bt^2 + Bt - \boxed{B} + Ct^3 + Dt^2 - Ct - \boxed{D}$$

t^3: $0 = A + B + C$ — $(+)$ $\quad 0 = 2A + 2B$ — $(+)$ $\quad 1 = 4A \Rightarrow A = 1/4$

t^2: $1 = A - B + D$ — $(+)$ $\quad 1 = 2A - 2B$ $\quad B = -1/4$

t: $0 = A + B - C$ $\quad C = 0$

1: $0 = A - B - D$ $\quad D = \frac{1}{2}$

$$I = 4\int\left(\frac{1/4}{t-1} + \frac{-1/4}{t+1} + \frac{1/2}{t^2+1}\right)dt = \ln|t-1| - \ln|t+1| + 2\,\mathrm{arctg}\,t + C$$

$$= \ln\left|\frac{\sqrt{\frac{1-x}{1+x}} - 1}{\sqrt{\frac{1-x}{1+x}} + 1}\right| + 2\,\mathrm{arctg}\sqrt{\frac{1-x}{1+x}} + C$$

$$\frac{\frac{\sqrt{1-x} - \sqrt{1+x}}{\cancel{\sqrt{1+x}}}}{\frac{\sqrt{1-x} + \sqrt{1+x}}{\cancel{\sqrt{1+x}}}} \cdot \frac{\sqrt{1-x} + \sqrt{1+x}}{\sqrt{1-x} - \sqrt{1+x}} = \frac{\cancel{1-x} - 2\sqrt{1-x^2} + \cancel{1+x}}{1-x-1+x}$$

$$= \frac{\sqrt{1-x^2} - 1}{x}$$

(3) 2. $I = \int\sqrt{\frac{x-1}{x+1}}\,dx \qquad a = 1, b = -1, c = 1, d = 1, \quad ad = 1 \neq -1 = bc$

$m = 2$

$$\frac{x-1}{x+1} = t^2, \quad x - 1 = xt^2 + t^2; \quad x(t^2-1) = -(t^2+1); \quad x = -\frac{t^2+1}{t^2-1}$$

$$dx = -\frac{2t(t^2-1) - (t^2+1)\cdot 2t}{(t^2-1)^2}\,dt = -\frac{\cancel{2t^3} - 2t - \cancel{2t^3} - 2t}{(t^2-1)^2}\,dt = \frac{4t\,dt}{(t^2-1)^2}$$

$$I = \int \sqrt{t^2}\,\frac{4t\,dt}{(t^2-1)^2} = 4\int \frac{t^2\,dt}{(t-1)^2(t+1)^2}$$

$$\frac{t^2}{(t-1)^2(t+1)^2} = \frac{A}{t-1}+\frac{B}{(t-1)^2}+\frac{C}{t+1}+\frac{D}{(t+1)^2} \quad /(t-1)^2(t+1)^2$$

$$t^2 = A\underbrace{(t-1)(t+1)^2}_{\substack{(t^2-1)(t+1)\\ =t^3+t^2-t-1}} + B\underbrace{(t+1)^2}_{t^2+2t+1} + C\underbrace{(t-1)^2(t+1)}_{\substack{(t^2-1)(t-1)\\ =t^3-t^2-t+1}} + D\underbrace{(t-1)^2}_{t^2-2t+1}$$

$$t^2 = At^3 + At^2 - At \boxed{-A} + Bt^2 + 2Bt \boxed{+B} + Ct^3 - Ct^2 - Ct \boxed{+C} + Dt^2 - 2Dt \boxed{+D}$$

$$\begin{aligned}
t^3:&\quad 0 = A + C \Rightarrow C = -A \\
t^2:&\quad 1 = A + B - C + D \\
t:&\quad 0 = -A + 2B - C - 2D \\
1:&\quad 0 = -A + B + C + D
\end{aligned}
\qquad
\begin{aligned}
1 &= 2A + B + D \quad —(+) \\
0 &= 2B - 2D \\
0 &= -2A + B + D
\end{aligned}$$

$$1 = 2B + 2D \quad —(+) \qquad 4B = 1 \Rightarrow B = 1/4,\ D = 1/4$$

$$0 = 2B - 2D \qquad A = \frac{B+D}{2} = 1/4, \quad C = -1/4$$

$$I = \int\left(\frac{1/4}{t-1} + \frac{1/4}{(t-1)^2} + \frac{-1/4}{t+1} + \frac{1/4}{(t+1)^2}\right)dt =$$

$$= \frac{1}{4}\left(\ln|t-1| - \frac{1}{t-1} - \ln|t+1| - \frac{1}{t+1}\right) + C$$

$$= \frac{1}{4}\left(\ln\left|\frac{t-1}{t+1}\right| - \frac{t+1-t+1}{t^2-1}\right) + C$$

$$\frac{x-1-x-1}{x+1} = \frac{-2}{x+1}$$

$$= \frac{1}{4}\ln\left|\underbrace{\frac{\sqrt{\frac{x-1}{x+1}}-1}{\sqrt{\frac{x-1}{x+1}}+1}}\right| - \frac{1}{2}\,\frac{1}{\frac{x-1}{x+1}-1} + C \;(=)$$

$$= \frac{\sqrt{x-1}-\sqrt{x+1}}{\sqrt{x-1}+\sqrt{x+1}}\cdot\frac{\sqrt{x-1}-\sqrt{x+1}}{\sqrt{x-1}-\sqrt{x+1}} = \frac{x-1+x+1-2\sqrt{x^2-1}}{x-1-x-1} = \sqrt{x^2-1}-x$$

$$(=)\ \frac{1}{4}\left(\ln\left|\sqrt{x^2-1}-x\right| + x + 1\right) + C$$

(1) 3. $I=\int \frac{1}{(1+x)^2}\sqrt{\frac{1+x}{1-x}}\,dx$ $\qquad a=1,\ b=1,\ c=-1,\ d=1;\ ad=1\neq-1=bc$

$m=2$

$$\frac{1+x}{1-x}=t^2;\quad 1+x=t^2-xt^2;\quad x(1+t^2)=t^2-1;\quad x=\frac{t^2-1}{t^2+1}$$

$$dx=\frac{2t(t^2+1)-(t^2-1)2t}{(t^2+1)^2}\,dt=\frac{\cancel{2t^3}+2t\cancel{-2t^3}+2t}{(t^2+1)^2}\,dt=\frac{4t\,dt}{(t^2+1)^2}$$

$$(1+x)^2=\left(1+\frac{t^2-1}{t^2+1}\right)^2=\left(\frac{t^2\cancel{+1}+t^2\cancel{-1}}{t^2+1}\right)^2=\frac{4t^4}{(t^2+1)^2}$$

$$I=\int\frac{1}{\frac{\cancel{4}t^{\cancel{4}2}}{\cancel{(t^2+1)^2}}}\cdot\cancel{\sqrt{t^2}}\cdot\frac{\cancel{4t}\,dt}{\cancel{(t^2+1)^2}}=\int t^{-2}\,dt=-\frac{1}{t}+C=$$

$$=-\frac{1}{\sqrt{\frac{1+x}{1-x}}}+C=-\sqrt{\frac{1-x}{1+x}}+C$$

4. $I=\int\frac{dx}{(2-x)\sqrt{1-x}}$ $\qquad a=-1,\ b=1,\ c=0,\ d=1;\ ad=-1\neq0=bc$

$m=2$

$$1-x=t^2\Rightarrow x=1-t^2\Rightarrow dx=-2t\,dt$$

$$I=\int\frac{-2\cancel{t}\,dt}{(2-1+t^2)\cdot\cancel{\sqrt{t^2}}}=-2\int\frac{dt}{t^2+1}=-2\,\mathrm{arctg}\,t+C=$$

$$=-2\,\mathrm{arctg}\sqrt{1-x}+C$$

5. $I=\int\frac{x-1}{\sqrt[3]{2x+1}}\,dx$ $\qquad a=2,\ b=1,\ c=0,\ d=1;\ ad=2\neq0=bc$

$m=3$

$$2x+1=t^3\Rightarrow x=\frac{t^3-1}{2}\Rightarrow dx=\frac{3t^2}{2}\,dt$$

$$I=\int\frac{\frac{t^3-1}{2}-1}{\sqrt[3]{t^3}}\cdot\frac{3t^2}{2}\,dt=\int\frac{\frac{t^3-3}{2}}{t}\cdot\frac{3}{2}t^2\,dt=\frac{3}{4}\int(t^4-3t)\,dt$$

$$=\frac{3}{4}\left(\frac{t^5}{5}-3\frac{t^2}{2}\right)+C=\frac{3}{4}\left(\frac{(2x+1)^{5/3}}{5}-\frac{3}{2}(2x+1)^{2/3}\right)+C$$

$$=\frac{3}{20}(2x+1)^{5/3}-\frac{9}{8}(2x+1)^{2/3}+C$$