IV $\int R(x, \sqrt{ax^2+bx+c})\,dx$

- Трансформацијом квадратног тринома ax^2+bx+c у збир или разлику квадрата дати интеграл сводимо на један од следећа три интеграла:

1) $\int R^*(t, \sqrt{t^2+1})\,dt$

2) $\int R^*(t, \sqrt{t^2-1})\,dt$

3) $\int R^*(t, \sqrt{1-t^2})\,dt$

Ове интеграле решавамо сменама:

1) $t = tg\, s$ или $t = sh\, s$
2) $t = \frac{1}{\cos s}$ или $t = ch\, s$
3) $t = \sin s$ или $t = th\, s$

1. $$\int \sqrt{x^2+1}\,dx = \left\{ \begin{array}{l} x = tg\, t \\ dx = \frac{dt}{\cos^2 t} \end{array} \right\} = \int \underbrace{\sqrt{tg^2 t+1}}_{\frac{\sin^2 t}{\cos^2 t}+1 = \frac{\sin^2 t+\cos^2 t}{\cos^2 t} = \frac{1}{\cos^2 t}} \cdot \frac{dt}{\cos^2 t} = \int \frac{dt}{\cos^3 t}$$

$$= \left\{ \begin{array}{l} tg\frac{t}{2} = s, \quad dt = \frac{2ds}{1+s^2} \\ \cos t = \frac{1-s^2}{1+s^2} \end{array} \right\} = \int \frac{\frac{2ds}{1+s^2}}{\left(\frac{1-s^2}{1+s^2}\right)^3} = 2\int \frac{(1+s^2)^2}{(1-s^2)^3}\,ds = \ldots$$

2. начин: (боље)

$$\int \sqrt{x^2+1}\,dx = \left\{ \begin{array}{l} x = sh\, t \\ dx = ch\, t\, dt \end{array} \right\} = \int \underbrace{\sqrt{sh^2 t+1}}_{ch^2 t}\, ch\, t\, dt = \int ch^2 t\, dt =$$

$$= \int \left(\frac{e^t+e^{-t}}{2}\right)^2 dt = \frac{1}{4}\int (e^{2t}+2+e^{-2t})\,dt =$$

$$= \frac{1}{4}\left(\frac{1}{2}e^{2t} + 2t - \frac{1}{2}e^{-2t}\right) + C = \frac{1}{8}e^{2\,\mathrm{arsh}\,x} + \frac{1}{2}\mathrm{arsh}\,x - \frac{1}{8}e^{-2\,\mathrm{arsh}\,x} + C =$$

$$= \frac{1}{8}e^{2\ln(x+\sqrt{x^2+1})} + \frac{1}{2}\ln(x+\sqrt{x^2+1}) - \frac{1}{8}e^{-2\ln(x+\sqrt{x^2+1})} + C =$$

$$= \frac{1}{8}(x+\sqrt{x^2+1})^2 + \frac{1}{2}\ln(x+\sqrt{x^2+1}) - \frac{1}{8}\frac{1}{(x+\sqrt{x^2+1})^2} + C$$

2. $\int \sqrt{2x^2+3x+3}\,dx = \dots$

$$2x^2+3x+3 = 2\left(x^2 + \frac{3}{2}x + \frac{3}{2}\right) = 2\left(x^2 + 2\cdot\frac{3}{4}x + \frac{9}{16} - \frac{9}{16} + \frac{3}{2}\right) =$$

$$= 2\left(\left(x+\frac{3}{4}\right)^2 + \frac{15}{16}\right) = 2\cdot\frac{15}{16}\left(\frac{16}{15}\left(\frac{4x+3}{4}\right)^2 + 1\right) =$$

$$= \frac{15}{8}\left(\left(\frac{\cancel{4}}{\sqrt{15}}\cdot\frac{4x+3}{\cancel{4}}\right)^2 + 1\right)$$

$$\dots = \int\sqrt{\frac{15}{8}\left(\left(\frac{4x+3}{\sqrt{15}}\right)^2+1\right)}\,dx = \left\{\begin{aligned} \frac{4x+3}{\sqrt{15}} &= t \\ \frac{4dx}{\sqrt{15}} &= dt \end{aligned}\right\} = \int \frac{\sqrt{15}}{2\sqrt{2}}\sqrt{t^2+1}\cdot\frac{\sqrt{15}}{4}\,dt$$

$$= \frac{15}{8\sqrt{2}}\int\sqrt{t^2+1}\,dt \overset{\boxed{1}}{=} \dots =$$

$$= \frac{15}{8\sqrt{2}}\left(\frac{1}{8}(t+\sqrt{t^2+1})^2 + \frac{1}{2}\ln(t+\sqrt{t^2+1}) - \frac{1}{8}\frac{1}{(t+\sqrt{t^2+1})^2}\right) + C$$

$$= \frac{15}{8\sqrt{2}}\left(\frac{1}{8}\left(\frac{4x+3}{\sqrt{15}} + \sqrt{\left(\frac{4x+3}{\sqrt{15}}\right)^2+1}\right)^2 + \frac{1}{2}\ln\left(\frac{4x+3}{\sqrt{15}} + \sqrt{\left(\frac{4x+3}{\sqrt{15}}\right)^2+1}\right)\right.$$

$$\left. - \frac{1}{8}\frac{1}{\left(\frac{4x+3}{\sqrt{15}} + \sqrt{\left(\frac{4x+3}{\sqrt{15}}\right)^2+1}\right)^2}\right) + C$$

3. $$\int\frac{dx}{\sqrt{x^2+1}} = \left\{\begin{aligned} x &= \mathrm{sh}\,t \\ dx &= \mathrm{ch}\,t\,dt \end{aligned}\right\} = \int\frac{\mathrm{ch}\,t\,dt}{\sqrt{\underbrace{\mathrm{sh}^2t+1}_{=\mathrm{ch}^2t}}} = \int dt = t + C =$$

$$= \mathrm{arsh}\,x + C = \ln(x+\sqrt{x^2+1}) + C$$

овај боље интересантније

2. начин: $$\int\frac{dx}{\sqrt{x^2+1}} = \left\{\begin{aligned} x &= \mathrm{tg}\,t \\ dx &= \frac{dt}{\cos^2 t} \end{aligned}\right\} = \int\frac{\frac{dt}{\cos^2 t}}{\sqrt{\mathrm{tg}\,t+1}} = \int\frac{\frac{dt}{\cos^2 t}}{\sqrt{\frac{1}{\cos^2 t}}} =$$

$$= \int\frac{dt}{\cos t} = \dots = \ln\left|\mathrm{tg}\frac{t+\pi/2}{2}\right| + C = \ln\left|\mathrm{tg}\frac{\mathrm{arctg}\,x+\pi/2}{2}\right| + C$$

4. $\int \frac{dx}{\sqrt{x^2+x+1}} = \int \frac{dx}{\sqrt{\frac{3}{4}\left(\left(\frac{2x+1}{\sqrt{3}}\right)^2+1\right)}} = \left\{ \begin{array}{l} \frac{2x+1}{\sqrt{3}} = t \\ \frac{2dx}{\sqrt{3}} = dt \end{array} \right\} = \int \frac{\frac{\sqrt{3}}{2}dt}{\frac{\sqrt{3}}{2}\sqrt{t^2+1}} = \ldots$

$x^2+x+1 = \left(x+\frac{1}{2}\right)^2 + \frac{3}{4} = \frac{3}{4}\left(\left(\frac{2}{\sqrt{3}}\frac{2x+1}{2}\right)^2+1\right)$

$\boxed{3} = \ln\left(t+\sqrt{t^2+1}\right) + C = \ln\left(\frac{2x+1}{\sqrt{3}} + \sqrt{\left(\frac{2x+1}{\sqrt{3}}\right)^2+1}\right) + C$

5. $I = \int \frac{dx}{(x+1)^2\sqrt{x^2+2x+2}} = \int \frac{dx}{(x+1)^2\sqrt{(x+1)^2+1}} = \left\{ \begin{array}{l} x+1 = t \\ dx = dt \end{array} \right\} =$

$= \int \frac{dt}{t^2\sqrt{t^2+1}} = \left\{ \begin{array}{l} t = tg\, s \\ dt = \frac{ds}{\cos^2 s} \end{array} \right\} = \int \frac{\frac{ds}{\cos^2 s}}{tg^2 s\sqrt{tg^2 s+1}} = \int \frac{\cancel{\frac{ds}{\cos^2 s}}}{\frac{\sin^2 s}{\cancel{\cos^2 s}}\frac{1}{\cos s}} =$

$= \int \frac{\cos s\, ds}{\sin^2 s} = \left\{ \begin{array}{l} \sin s = z \\ \cos s\, ds = dz \end{array} \right\} = \int \frac{dz}{z^2} = -\frac{1}{z} + C =$

$= -\frac{1}{\sin s} + C = -\frac{1}{\sin(\operatorname{arctg} t)} + C = -\frac{1}{\sin(\operatorname{arctg}(x+1))} + C$

2. način:

$I = \ldots \int \frac{dt}{t^2\sqrt{t^2+1}} = \left\{ \begin{array}{l} t = sh\, s \\ dt = \cancel{ch\, s\, ds} \\ ch\, s\, ds \end{array} \right\} = \int \frac{\cancel{ch\, s}\, ds}{sh^2 s\sqrt{sh^2 s+1}} =$

$= \int \frac{ds}{\frac{e^{2s}+2+e^{-2s}}{4}} = \int \frac{4e^{2s}ds}{e^{4s}+2e^{2s}+1} = \int \frac{4e^{2s}ds}{(e^{2s}+1)^2} =$

$= \left\{ \begin{array}{l} e^{2s}+1 = z \\ 2e^{2s}ds = dz \end{array} \right\} = \int \frac{2dz}{z^2} = -\frac{2}{z} + C = -\frac{2}{e^{2s}+1} + C =$

$= -\frac{2}{e^{2\operatorname{arsh} t}+1} + C = -\frac{2}{\left(t+\sqrt{t^2+1}\right)^2+1} + C =$

$= -\frac{2}{\left(x+1+\sqrt{\underbrace{(x+1)^2+1}_{=x^2+2x+2}}\right)^2+1} + C$

6. $\int \sqrt{x^2-1}\, dx = \left| \begin{array}{l} x = \frac{1}{\cos t} \\ dx = +\frac{\sin t}{\cos^2 t} dt \end{array} \right| = \int \sqrt{\frac{1}{\cos^2 t} - 1}\ \frac{\sin t}{\cos^2 t} dt$

$= \int \frac{\sin t}{\cos t} \cdot \frac{\sin t}{\cos^2 t} dt = \int \frac{\sin^2 t\, dt}{\cos^3 t\, dt} = \left| \begin{array}{l} tg \frac{t}{2} = s,\ dt = \frac{2ds}{1+s^2} \\ \sin t = \frac{2s}{1+s^2},\ \cancel{\cos t = \frac{1+s^2}{1+s^2}} \\ \cos t = \frac{1-s^2}{1+s^2} \end{array} \right|$

$= \int \frac{\frac{4s^2}{\cancel{(1+s^2)^2}} \cdot \frac{2ds}{\cancel{1+s^2}}}{\left(\frac{1-s^2}{\cancel{1+s^2}}\right)^3} = 8 \int \frac{s^2\, ds}{(1-s^2)^3} = \ldots$

2. начин (боље):

$\int \sqrt{x^2-1}\, dx = \left| \begin{array}{l} x = ch\, t \\ dx = sh\, t\, dt \end{array} \right| = \int \sqrt{\underset{sh^2 t}{ch^2 t - 1}} \cdot sh\, t\, dt =$

$= \int sh^2 t\, dt = \int \left(\frac{e^t - e^{-t}}{2} \right)^2 dt = \frac{1}{4} \int (e^{2t} - 2 + e^{-2t}) dt =$

$= \frac{1}{4} \left(\frac{1}{2} e^{2t} - 2t - \frac{1}{2} e^{-2t} \right) + C =$

$= \frac{1}{8} e^{2 arch\, x} - \frac{1}{2} arch\, x - \frac{1}{8} e^{-2 arch\, x} + C =$

$= \frac{1}{8} e^{2 \ln (x + \sqrt{x^2-1})} - \frac{1}{2} \ln (x + \sqrt{x^2-1}) - \frac{1}{8} e^{-2 \ln (x + \sqrt{x^2-1})} + C =$

$= \frac{1}{8} (x + \sqrt{x^2-1})^2 - \frac{1}{2} \ln (x + \sqrt{x^2-1}) - \frac{1}{8} \frac{1}{(x + \sqrt{x^2-1})^2} + C$

$\cancel{\frac{1}{8}(}$

7. $\int\sqrt{2x^2-3}\,dx = \int\sqrt{3\left(\left(x\sqrt{\tfrac{2}{3}}\right)^2-1\right)}\,dx = \left\{\begin{array}{l} x\sqrt{\tfrac{2}{3}} = t \\ \sqrt{\tfrac{2}{3}}\,dx = dt \end{array}\right\} =$

$2x^2-3 = 3\left(\tfrac{2}{3}x^2-1\right) = 3\left(\left(x\sqrt{\tfrac{2}{3}}\right)^2-1\right)$

$= \int\sqrt{3}\sqrt{t^2-1}\cdot\sqrt{\tfrac{3}{2}}\,dt = \frac{3}{\sqrt{2}}\int\sqrt{t^2-1}\,dt = \boxed{6}$

$\ldots = \frac{3}{\sqrt{2}}\left(\frac{1}{8}\left(t+\sqrt{t^2-1}\right)^2 - \frac{1}{2}\ln\left(t+\sqrt{t^2-1}\right) - \frac{1}{8}\frac{1}{\left(t+\sqrt{t^2-1}\right)^2}\right)+C=$

$= \frac{3}{\sqrt{2}}\left(\frac{1}{8}\left(x\sqrt{\tfrac{2}{3}}+\sqrt{\tfrac{2x^2}{3}-1}\right)^2 - \frac{1}{2}\ln\left(x\sqrt{\tfrac{2}{3}}+\sqrt{\tfrac{2x^2}{3}-1}\right) - \frac{1}{8}\frac{1}{\left(x\sqrt{\tfrac{2}{3}}+\sqrt{\tfrac{2x^2}{3}-1}\right)^2}\right)+C$

8. $\int\frac{dx}{\sqrt{x^2-1}} = \left\{\begin{array}{l} x = \operatorname{ch}t \\ dx = \operatorname{sh}t\,dt \end{array}\right\} = \int\frac{\operatorname{sh}t\,dt}{\underbrace{\sqrt{\operatorname{ch}^2t-1}}_{=\operatorname{sh}^2t}} = \int dt = t+C =$

$= \operatorname{arch}x + C = \ln\left(x+\sqrt{x^2-1}\right)+C$

2. način:

$\int\frac{dx}{\sqrt{x^2-1}} = \left\{\begin{array}{l} x = \frac{1}{\cos t} \\ dx = \frac{\sin t}{\cos^2t}\,dt \end{array}\right\} = \int\frac{\frac{\sin t}{\cos^2t}\,dt}{\sqrt{\frac{1}{\cos^2t}-1}} = \int\frac{\frac{\sin t\,dt}{\cos^2t}}{\frac{\sin t}{\cos t}} =$

$= \int\frac{dt}{\cos t} = \ldots = \ln\left|\operatorname{tg}\frac{t+\pi/2}{2}\right|+C =$

$= \ln\left|\operatorname{tg}\frac{\arccos\frac{1}{x}+\pi/2}{2}\right|+C$

9. $\int\frac{dx}{\sqrt{x^2-3x+1}} = \int\frac{dx}{\sqrt{\frac{5}{4}\left(\left(\frac{2x-3}{\sqrt{5}}\right)^2-1\right)}} = \left\{\begin{array}{l} \frac{2x-3}{\sqrt{5}} = t \\ \frac{2dx}{\sqrt{5}} = dt \end{array}\right\} = \int\frac{\frac{\sqrt{5}}{2}\,dt}{\frac{\sqrt{5}}{2}\sqrt{t^2-1}} = \ldots$

$x^2-3x+1 = x - 2\cdot\frac{3}{2}x + \frac{9}{4} - \frac{9}{4} + 1 = \left(x-\frac{3}{2}\right)^2 - \frac{5}{4} = \frac{5}{4}\left(\left(\frac{2}{\sqrt{5}}\frac{2x-3}{2}\right)^2-1\right)$

$\boxed{8} = \ln\left(t+\sqrt{t^2-1}\right)+C = \ln\left(\frac{2x-3}{\sqrt{5}}+\sqrt{\left(\frac{2x-3}{\sqrt{5}}\right)^2-1}\right)+C$

10. $\int\sqrt{1-x^2}\,dx = \left|\begin{matrix} x=\sin t\\ dx=\cos t\,dt\end{matrix}\right| = \int \underbrace{\sqrt{1-\sin^2 t}}_{=\cos t}\cos t\,dt = \int\cos^2 t\,dt$

$= \int \frac{1+\cos 2t}{2}\,dt = \frac{1}{2}\left(t+\frac{1}{2}\sin 2t\right)+C =$

$= \frac{1}{2}\arcsin x + \frac{1}{4}\sin(2\arcsin x)+C$

2. начин:

$\int\sqrt{1-x^2}\,dx = \left|\begin{matrix}x=\operatorname{th} t\\ dx=\frac{dt}{\operatorname{ch}^2 t}\end{matrix}\right| = \int\sqrt{1-\operatorname{th}^2 t}\,\frac{dt}{\operatorname{ch}^2 t} = \int\frac{dt}{\operatorname{ch}^3 t}=$

$=\int\frac{dt}{\left(\frac{e^t+e^{-t}}{2}\right)^3} = 8\int\frac{dt}{\left(\frac{e^{2t}+1}{e^t}\right)^3} = 8\int\frac{e^{2t}\cdot e^t\,dt}{(e^{2t}+1)^3} = \left|\begin{matrix}e^t=s\\ e^t dt=ds\end{matrix}\right| =$

$= 8\int\frac{t^2\,dt}{(t^2+1)^3} = \dots$

11. $\int\sqrt{3-2x-x^2}\,dx = \int\sqrt{4\left(1-\left(\frac{x+1}{2}\right)^2\right)}\,dx = \left|\begin{matrix}\frac{x+1}{2}=t\\ \frac{dx}{2}=dt\end{matrix}\right| = \int 2\sqrt{1-t^2}\,2dt = \dots$

$3-2x-x^2 = -(x^2+2x-3) = -((x+1)^2-4) = -4\left(\left(\frac{x+1}{2}\right)^2-1\right)$

$=4\left(1-\left(\frac{x+1}{2}\right)^2\right)$

$\overset{10}{=} 4\cdot\left(\frac{1}{2}\arcsin t + \frac{1}{4}\sin(2\arcsin t)\right)+C =$

$= 2\arcsin\cancel{t}\,\frac{x+1}{2} + \sin\left(2\arcsin\frac{x+1}{2}\right)+C$

12. $\int\sqrt{x-4x^2} = \int\sqrt{\frac{1}{16}\left(1-(8x-1)^2\right)}\,dx = \left|\begin{matrix}8x-1=t\\ 8dx=dt\end{matrix}\right| = \frac{1}{4}\int\sqrt{1-t^2}\cdot\frac{dt}{8} = \dots$

$x-4x^2 = -4\left(x^2-\frac{1}{4}x\right) = -4\left(x^2-2\cdot\frac{1}{8}x+\frac{1}{64}-\frac{1}{64}\right) = -4\cdot\frac{1}{64}\left(64\left(x-\frac{1}{8}\right)^2-1\right)$

$= -\frac{1}{16}\left(\left(8\cdot\frac{8x-1}{8}\right)^2-1\right) = \frac{1}{16}\left(1-(8x-1)^2\right)$

$\overset{\boxed{10}}{=}\frac{1}{32}\left(\frac{1}{2}\arcsin t+\frac{1}{4}\sin(2\arcsin t)\right)+C =$

$=\frac{1}{64}\arcsin(8x-1)+\frac{1}{128}\sin(2\arcsin t)+C$