III Ојлерове смене

$$I=\int R(x,\sqrt{ax^2+bx+c})\,dx$$

1) $ax^2+bx+c = a(x-x_1)(x-x_2)$, $x_1, x_2 \in \mathbb{R}$: $\quad a(x-x_1)=(x-x_2)t^2$

$x=\dots,\ dx=\dots$

2) $a>0$: $\sqrt{ax^2+bx+c} = t \pm \sqrt{a}\,x$

$x=\dots,\ dx=\dots$

3) $c>0$: $\sqrt{ax^2+bx+c} = tx \pm \sqrt{c}$

$x=\dots,\ dx=\dots$

$$I=\int R^*(t)\,dt$$

1. $I=\int \frac{dx}{(x-1)\sqrt{x^2-3x+2}}$

$x^2-3x+2=(x-2)(x-1)$, $2, 1 \in \mathbb{R}$ $\Rightarrow$ може 1)

$a=1>0 \Rightarrow$ може 2)

$c=2>0 \Rightarrow$ може 3)

$x-2=(x-1)t^2$, $x-2=xt^2-t^2$, $x(t^2-1)=t^2-2$, $x=\frac{t^2-2}{t^2-1}$

$$dx=\frac{2t(t^2-1)-(t^2-2)\cdot 2t}{(t^2-1)^2}dt=\frac{\cancel{2t^3}-2t-\cancel{2t^3}+4t}{(t^2-1)^2}dt=\frac{2t\,dt}{(t^2-1)^2}$$

$$I=\int\frac{\frac{2t\,dt}{(t^2-1)^2}}{\left(\frac{t^2-2}{t^2-1}-1\right)\cdot\left(\frac{t^2-2}{t^2-1}-1\right)t}=2\int\frac{\frac{dt}{\cancel{(t^2-1)^2}}}{\left(\frac{\cancel{t^2}-2-\cancel{t^2}+1}{\cancel{t^2-1}}\right)^2}=2\int dt=\dots$$

$$\sqrt{x^2-3x+2}=\sqrt{(x-2)(x-1)}=\cancel{\sqrt{(x-2)(x-1)}}=\sqrt{(x-1)t^2(x-1)}=\sqrt{(x-1)^2t^2}=(x-1)t$$

$$\dots=2t+C=2\sqrt{\frac{x-2}{x-1}}+C$$

$$2.\ I=\int \frac{dx}{1-x^2+2\sqrt{1-x^2}}$$

сп

$1-x^2=-(x+1)(x-1) \Rightarrow$ може (1)

$a=-1<0 \Rightarrow$ не може 2)

$c=1>0 \Rightarrow$ може 3)

$$-(x+1)=(x-1)t^2;\quad -x-1=xt^2-t^2;\quad x(t^2+1)=t^2-1,\quad x=\frac{t^2-1}{t^2+1}$$

$$dx=\frac{2t(t^2+1)-(t^2-1)2t}{(t^2+1)^2}dt=\frac{\cancel{2t^3}+2t-\cancel{2t^3}+2t}{(t^2+1)^2}dt=\frac{4t\,dt}{(t^2+1)^2}$$

$$I=\int\frac{\frac{4t\,dt}{(t^2+1)^2}}{1-\left(\frac{t^2-1}{t^2+1}\right)^2+2\left(\frac{t^2-1}{t^2+1}-1\right)t}=4\int\frac{\frac{t\,dt}{(t^2+1)^{\cancel{2}}}}{\frac{\cancel{t^4}+2t^2+\cancel{1}-\cancel{t^4}+2t^2-\cancel{1}}{(t^2+1)^{\cancel{2}}}+2\frac{\cancel{t^2}-1-\cancel{t^2}-1}{\cancel{t^2+1}}t}$$

$$\sqrt{1-x^2}=\sqrt{-(x+1)(x-1)}=\sqrt{(x-1)t^2\cdot(x-1)}=\sqrt{(x-1)^2t^2}=(x-1)t$$

$$\dots=\cancel{4}\int\frac{\frac{\cancel{t}\,dt}{t^2+1}}{\frac{\cancel{4}t^{\cancel{2}}}{t^2+1}-\cancel{4t}}=\int\frac{\frac{dt}{\cancel{t^2+1}}}{\frac{t-t^2-1}{\cancel{t^2+1}}}=-\int\frac{dt}{t^2-t+1}=\dots$$

$$t^2-t+1=\left(t-\frac12\right)^2+\frac34=\frac34\left(\left(\frac{\cancel{2}}{\sqrt3}\cdot\frac{2t-1}{\cancel{2}}\right)^2+1\right)$$

$$\dots=-\int\frac{dt}{\frac34\left(\left(\frac{2t-1}{\sqrt3}\right)^2+1\right)}=\left\{\begin{aligned}\frac{2t-1}{\sqrt3}&=s\\ \frac{2dt}{\sqrt3}&=ds\end{aligned}\right\}=-\int\frac{\frac{\sqrt3}{2}ds}{\frac34(s^2+1)}=$$

$$=-\frac{2}{\sqrt3}\operatorname{arctg}s+C=-\frac{2}{\sqrt3}\operatorname{arctg}\frac{2t-1}{\sqrt3}+C=$$

$$=-\frac{2}{\sqrt3}\operatorname{arctg}\frac{2\cdot\sqrt{\frac{1+x}{1-x}}-1}{\sqrt3}+C$$

3. $I = \int \frac{dx}{x\sqrt{x^2+2x+3}}$

$x^2+2x+3=0 \Rightarrow x_{1/2} = \frac{-2\pm\sqrt{4-12}}{2} \notin \mathbb{R} \Rightarrow$ не може 1)

$a=1>0 \Rightarrow$ може (2)

$c=3>0 \Rightarrow$ може 3)

$\sqrt{x^2+2x+3} = t+\sqrt{1}\,x \;/^2$

$\cancel{x^2}+2x+3 = t^2+2tx+\cancel{x^2}$

$2x(1-t) = t^2-3$

$x = \frac{t^2-3}{2(1-t)}$

$dx = \frac{1}{2}\cdot\frac{2t(1-t)-(t^2-3)(-1)}{(1-t)^2}dt$

$= \frac{2t-2t^2+t^2-3}{2(1-t)^2}dt$

$= \frac{-(t^2-2t+3)}{2(1-t)^2}dt$

$$I = \int \frac{\frac{-(t^2-2t+3)}{\cancel{2}(1-t)^{\cancel{2}}}dt}{\frac{t^2-3}{\cancel{2(1-t)}}\left(t+\frac{t^2-3}{2(1-t)}\right)} = \int \frac{\frac{\cancel{-(t^2-2t+3)}}{\cancel{1-t}}dt}{(t^2-3)\frac{\cancel{2t-2t^2+t^2-3}}{2\cancel{(1-t)}}} =$$

$$= 2\int \frac{dt}{t^2-3}$$

$\frac{1}{t^2-3} = \frac{1}{(t-\sqrt{3})(t+\sqrt{3})} = \frac{A}{t-\sqrt{3}} + \frac{B}{t+\sqrt{3}} \;/(t-\sqrt{3})(t+\sqrt{3})$

$1 = A(t+\sqrt{3}) + B(t-\sqrt{3})$

$t: \quad 0 = A+B \;/\cdot\sqrt{3}$

$1: \quad 1 = A\sqrt{3} - B\sqrt{3}$ $\Big\}(+)$

$1 = 2A\sqrt{3} \Rightarrow A = 1/(2\sqrt{3})$

$B = -1/(2\sqrt{3})$

$$I = 2\int\left(\frac{1/(2\sqrt{3})}{t-\sqrt{3}} + \frac{-1/(2\sqrt{3})}{t+\sqrt{3}}\right)dt = \frac{1}{\sqrt{3}}\ln\left|\frac{t-\sqrt{3}}{t+\sqrt{3}}\right| + C =$$

$$= \frac{1}{\sqrt{3}}\ln\left|\frac{\sqrt{x^2+2x+3}-x-\sqrt{3}}{\sqrt{x^2+2x+3}-x+\sqrt{3}}\right| + C$$

4. $\cancel{\int \frac{x^2 dx}{\sqrt{9+x^2}}}$ $I = \int \sqrt{x^2-2x+2}$ (овде преко Ојлера, али боље преко хиперболичних)

x^2-2x+2 нерастављив $\Rightarrow$ не може 1)

$a = 1 > 0 \Rightarrow$ може 2)

$c = 2 > 0 \Rightarrow$ може 3)

$$\sqrt{x^2-2x+2} = t - \sqrt{1}\, x \quad /^2$$

$$\cancel{x^2} - 2x + 2 = t^2 - 2tx + \cancel{x^2}$$

$$2x(t-1) = t^2 - 2$$

$$x = \frac{t^2-2}{2(t-1)}$$

$$dx = \frac{1}{2} \frac{2t(t-1) - (t^2-2)\cdot 1}{(t-1)^2} dt = \frac{2t^2-2t-t^2+2}{2(t-1)^2} dt = \frac{t^2-2t+2}{2(t-1)^2} dt$$

$$I = \int \left(t - \frac{t^2-2}{2(t-1)}\right) \cdot \frac{t^2-2t+2}{2(t-1)^2} dt = \int \frac{2t^2-2t-t^2+2}{2(t-1)} \cdot \frac{t^2-2t+2}{2(t-1)^2} dt$$

$$= \frac{1}{4} \int \frac{(t^2-2t+2)^2}{(t-1)^3} dt = \frac{1}{4} \int \frac{t^4-4t^3+8t^2-8t+4}{t^3-3t^2+3t-1} dt$$

$$(t^2-2t+2)^2 = t^4 + 4t^2 + 4 - 4t^3 + 4t^2 - 8t$$

$$(t^4-4t^3+8t^2-8t+4) : (t^3-3t^2+3t-1) = t - 1 + \frac{2t^2-4t+3}{(t-1)^3}$$

$$\begin{array}{l} t^4 - 3t^3 + 3t^2 - t \\ \hline - \quad -t^3 + 5t^2 - 7t + 4 \\ \quad -t^3 + 3t^2 - 3t + 1 \\ \hline - \quad 2t^2 - 4t + 3 \end{array}$$

$$I = \frac{1}{4} \int \left(t - 1 + \frac{2t^2-4t+3}{(t-1)^3}\right) dt$$

$$\frac{2t^2-4t+3}{(t-1)^3} = \frac{A}{t-1} + \frac{B}{(t-1)^2} + \frac{C}{(t-1)^3} \quad / (t-1)^3$$

$$2t^2-4t+3 = A(t^2-2t+1) + B(t-1) + C$$

$t^2: \quad 2 = A$

$t: \quad -4 = -2A + B \Rightarrow B = 0$

$1: \quad 3 = A - B + C \Rightarrow C = 1$

$$I = \frac{1}{4}\int\left(t - 1 + \frac{2}{t-1} + \frac{0}{(t-1)^2} + \frac{1}{(t-1)^3}\right)dt =$$

$$= \frac{1}{4}\left(\frac{t^2}{2} - t + 2\ln|t| - \frac{1}{2(t-1)^2}\right) + C$$

$$= \frac{1}{4}\left(\frac{\left(\sqrt{x^2-2x+2} + x\right)^2}{2} - \sqrt{x^2-2x+2} - x + 2\ln\left|\sqrt{x^2-2x+2} + x\right|\right.$$

$$\left. - \frac{1}{2\left(\sqrt{x^2-2x+2} + x - 1\right)^2}\right) + C$$

5. $I = \int \frac{dx}{(1+x)\sqrt{1-x-x^2}}$

$1 - x - x^2 = 0, \quad x_{1/2} = \frac{1 \pm \sqrt{1+4}}{-2} = -\frac{1}{2} \pm \frac{\sqrt{5}}{2} \in \mathbb{R} \Rightarrow$ može 1)

$a = -1 < 0 \Rightarrow$ ne može 2)

$c = 1 > 0 \Rightarrow$ može (3)

$\sqrt{1-x-x^2} = tx + \sqrt{1} \quad /^2$

$\cancel{1} - x - x^2 = t^2x^2 + 2tx + \cancel{1} \quad /:x$

$-1 - x = t^2x + 2t$

$t^2x + x = -2t - 1$

$(t^2 + 1)x = -(2t+1)$

$x = -\frac{2t+1}{t^2+1}$

$$dx = -\frac{2(t^2+1) - (2t+1)2t}{(t^2+1)^2}dt$$

$$= -\frac{2t^2 + 2 - 4t^2 - 2t}{(t^2+1)^2}dt$$

$$= \frac{2(t^2+t-1)}{(t^2+1)^2}dt$$

$$I = \int \frac{\frac{2(t^2+t-1)}{(t^2+1)^2}dt}{\left(1 - \frac{2t+1}{t^2+1}\right)\cdot\left(t\left(-\frac{2t+1}{t^2+1}\right) + 1\right)} = \int \frac{\frac{2\cancel{(t^2+t-1)}^{-1}}{\cancel{(t^2+1)^2}}dt}{\frac{t^2+1-2t-1}{\cancel{t^2+1}}\cdot\frac{\cancel{-2t^2-t+t^2+1}}{\cancel{t^2+1}}}$$

$$= -2\int \frac{dt}{t(t-2)}$$

$$\frac{1}{t(t-2)} = \frac{A}{t} + \frac{B}{t-2} \Big/ \, t(t-2)$$

$$1 = A(t-2) + Bt$$

$$t: \quad 0 = A + B \quad \Rightarrow B = 1/2$$

$$1: \quad 1 = -2A \quad \Rightarrow A = -1/2$$

$$I = -2\int \left(\frac{-1/2}{t} + \frac{1/2}{t+2} \right) dt = -\cancel{2} \cdot \frac{1}{\cancel{2}} \ln \left| \frac{t+2}{t} \right| + C =$$

$$= \ln \left| \frac{t}{t+2} \right| + C = \ln \left| \frac{\frac{\sqrt{1-x-x^2}-1}{x}}{\frac{\sqrt{1-x-x^2}-1}{x} + 2} \right| + C =$$

$$= \ln \left| \frac{\sqrt{1-x-x^2}-1}{\sqrt{1-x-x^2}-1+2x} \right| + C$$