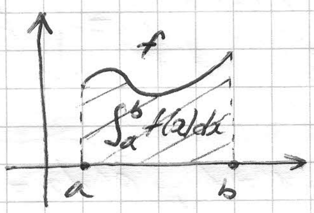
ОДРЕЂЕНИ ИНТЕГРАЛ

- $\int_a^b f(x)\,dx$



- Особине:

⊛ $\int_a^b \alpha f(x)\,dx = \alpha \int_a^b f(x)\,dx, \quad \alpha \in \mathbb{R}$ — хомогеност

⊛ $\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

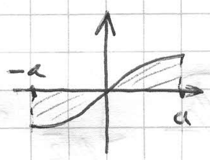
— адитивност по интегранду

⊛ $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

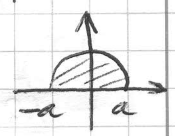
— адитивност по области интеграције

⊛ $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

⊛ f непарна $(f(-x) = -f(x))$: $\int_{-a}^{a} f(x)\,dx = 0$



f парна $(f(-x) = f(x))$: $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$



Њутн-Лајбницова формула

$$\int_a^b f(x)\,dx = [F(x)]\Big|_a^b = F(b) - F(a),$$

F — једна примитивна ф-ја ф-је f

Пр. $\int_{\pi/4}^{\pi/2} \cos x\,dx = [\sin x]\Big|_{\pi/4}^{\pi/2} = \sin\frac{\pi}{2} - \sin\frac{\pi}{4} = 1 - \frac{1}{\sqrt{2}}$

(нема константе)

Смена променљиве

$$\int_a^b f(x)\,dx = \left\{ \begin{array}{l} x=\varphi(t),\quad dx=\varphi'(t)\,dt \\ a\to\alpha,\quad b\to\beta \end{array} \right\} = \int_\alpha^\beta f(\varphi(t))\,\varphi'(t)\,dt$$

Пр. $\int_0^1 (e^x-1)^4 e^x\,dx = \left\{ \begin{array}{l} e^x-1=t,\quad e^x dx = dt \\ x=0 \Rightarrow t=0,\quad x=1 \Rightarrow t=e-1 \end{array} \right\} =$

$$= \int_0^{e-1} t^4\,dt = \left[\frac{t^5}{5}\right]\Bigg|_0^{e-1} = \frac{(e-1)^5}{5} - \cancel{\frac{0^5}{5}}$$

(нема враћања смене)

Парцијална интеграција

$$\int_a^b u\,dv = [uv]\Big|_a^b - \int_a^b v\,du$$

Пр. $\int_{1/\sqrt{2}}^1 \arcsin x\,dx = \left\{ \begin{array}{l} u=\arcsin x,\quad dv=dx \\ du=\frac{dx}{\sqrt{1-x^2}},\quad v=x \end{array} \right\} =$

$$[x\arcsin x]\Big|_{1/\sqrt{2}}^1 - \int_{1/\sqrt{2}}^1 \frac{x\,dx}{\sqrt{1-x^2}} = \left\{ \begin{array}{l} 1-x^2=t,\quad -2x\,dx=dt \\ x=1/\sqrt{2} \Rightarrow t=1/2 \\ x=1 \Rightarrow t=0 \end{array} \right\} =$$

$$1\cdot\frac{\pi}{2} - 0\cdot 0 - \int_{1/2}^0 \frac{\frac{dt}{-2}}{\sqrt{t}} = \frac{\pi}{2} - \int_0^{1/2} t^{-1/2}\,dt =$$

$$= \frac{\pi}{2}\left[\frac{t^{1/2}}{1/2}\right]\Bigg|_0^{1/2} = \pi\left(\frac{1}{\sqrt{2}} - 0\right) = \frac{\pi}{\sqrt{2}}$$

1. $I = \int_9^{16} \frac{dx}{\sqrt{x+9} - \sqrt{x}} = \int_9^{16} \frac{1}{\sqrt{x+9} - \sqrt{x}} \, \frac{\sqrt{x+9} + \sqrt{x}}{\sqrt{x+9} + \sqrt{x}} \, dx =$

$= \int_9^{16} \frac{\sqrt{x+9} + \sqrt{x}}{\cancel{x}+9-\cancel{x}} \, dx = \frac{1}{9} \int_9^{16} \left((x+9)^{1/2} + x^{1/2} \right) dx =$

$= \frac{1}{9} \left[\frac{2}{3} (x+9)^{3/2} + \frac{2}{3} x^{3/2} \right] \Big|_9^{16} = \frac{2}{27} \left(25^{3/2} - 18^{3/2} + 16^{3/2} - 9^{3/2} \right)$

$= \frac{2}{27} \left(125 - 54\sqrt{2} + 64 - 27 \right) = \frac{2}{\cancel{27}} \left(\overset{6}{\cancel{162}} - \overset{2}{\cancel{54}}\sqrt{2} \right) = 4(3 - \sqrt{2})$

2. $I = \int_0^1 \frac{x \, dx}{(x+1)(x+2)(x+3)}$

$\frac{x}{(x+1)(x+2)(x+3)} = \frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{x+3} \quad / (x+1)(x+2)(x+3)$

$x = A(x+2)(x+3) + B(x+1)(x+3) + C(x+1)(x+2)$

$= Ax^2 + 5Ax + 6A + Bx^2 + 4Bx + 3B + Cx^2 + 3Cx + 2C$

$x^2: \quad 0 = A + B + C \Rightarrow C = -A - B$

$x: \quad 1 = 5A + 4B + 3C$

$1: \quad 0 = 6A + 3B + 2C$

$1 = 2A + B \quad / \cdot (-1)$

$0 = 4A + B \quad (+)$

$-1 = 2A \Rightarrow A = -1/2$

$B = 2$

$C = -3/2$

$I = \int_0^1 \left(\frac{-1/2}{x+1} + \frac{2}{x+2} + \frac{-3/2}{x+3} \right) dx =$

$= -\frac{1}{2} \ln|x+1| + 2 \ln|x+2| - \frac{3}{2} \ln|x+3|$

$= \ln \frac{(x+2)^2}{(x+1)^{1/2} (x+3)^{3/2}}$

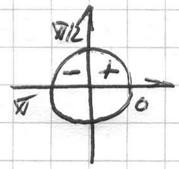
(4) 3. $I = \int_0^{T/2} \sin\left(\frac{2\pi t}{T} - \varphi\right) dt = \left| \begin{matrix} \frac{2\pi t}{T} - \varphi = s \\ \frac{2\pi}{T} dt = ds \\ t = 0 \rightarrow s = -\varphi \\ t = \frac{T}{2} \rightarrow s = \pi - \varphi \end{matrix} \right| =$

$= \int_{-\varphi}^{\pi-\varphi} \sin s \cdot \frac{T}{2\pi} ds$

$= \frac{T}{2\pi} [-\cos s] \Big|_{-\varphi}^{\pi-\varphi} = -\frac{T}{2\pi} \left(\underbrace{\cos(\pi - \varphi)}_{= -\cos\varphi} - \underbrace{\cos(-\varphi)}_{= \cos\varphi} \right) =$

$= -\frac{T}{\cancel{2}\pi} (-\cancel{2}\cos\varphi) = \frac{T\cos\varphi}{\pi}$

(3) 4. $I = \int_0^{\pi} \sqrt{\frac{1+\cos 2x}{2}}\, dx = \int_0^{\pi} \sqrt{\cos^2 x}\, dx = \int_0^{\pi} |\cos x|\, dx =$



$= \int_0^{\pi/2} |\cos x|\, dx + \int_{\pi/2}^{\pi} |\cos x|\, dx =$

$= \int_0^{\pi/2} \cos x\, dx + \int_{\pi/2}^{\pi} (-\cos x)\, dx =$

$= [\sin x]\Big|_0^{\pi/2} - [\sin x]\Big|_{\pi/2}^{\pi} = (1-0) - (0-1) = 2$

(Грешно: $I = \int_0^{\pi} \cos x\, dx = [\sin x]\Big|_0^{\pi} = 0 - 0 = 0$)

5. $I = \int_4^9 \frac{\sqrt{x}}{\sqrt{x} - 1} dx = \left| \begin{matrix} x = t^2, \ dx = 2t\,dt \\ x = 4 \rightarrow t^2 = 4 \rightarrow t = \pm 2 \\ x = 9 \rightarrow t^2 = 9 \rightarrow t = \pm 3 \end{matrix} \right|$

(1)

$\downarrow$

[illegible]

$I = \int_2^3 \frac{\sqrt{t^2}}{\sqrt{t^2} - 1} 2t\,dt = 2\int_2^3 \frac{|t|}{|t| - 1} t\,dt = \left| \begin{matrix} t\Big|_2^3, \\ |t| = t \end{matrix} \right|$

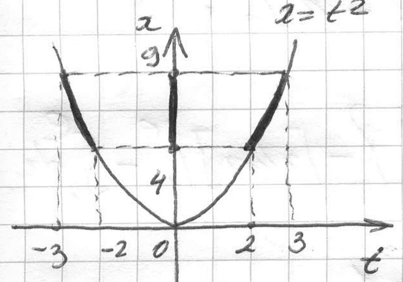
$= 2\int_2^3 \frac{t}{t-1} t\,dt = 2\int_2^3 \frac{t^2}{t-1} dt =$, $t^2 = t^2 - 1 + 1 = (t-1)(t+1) + 1$

$$= 2\int_2^3 \left(t+1+\frac{1}{t-1}\right)dt = 2\left[\frac{t^2}{2}+t+\ln|t-1|\right]\Big|_2^3 =$$

$$= 2\left(\frac{9}{2}+3+\ln 2-2-2-\ln 1\right) = 7+2\ln 2$$

(2) $I=\int_{2}^{-3}\frac{\sqrt{t^2}}{\sqrt{t^2}-1}\,2t\,dt$; (3) $I=\int_{-2}^{3}\frac{\sqrt{t^2}}{\sqrt{t^2}-1}\,2t\,dt$; (4) $I=\int_{-2}^{-3}\frac{\sqrt{t^2}}{\sqrt{t^2}-1}\,2t\,dt$

$\downarrow$ неверно ; $\downarrow$ неверно ; $\downarrow$ верно



6. $I=\int_0^{\pi/2} x^2\sin 3x\,dx = \left\{\begin{matrix} u=x^2, & dv=\sin 3x\,dx \\ du=2x\,dx, & v=-\frac{1}{3}\cos 3x \end{matrix}\right\} =$

$$= \underbrace{\left[-\frac{1}{3}x^2\cos 3x\right]\Big|_0^{\pi/2}}_{=0} + \frac{2}{3}\int x\cos 3x\,dx = \left\{\begin{matrix} u=x, & dv=\cos 3x\,dx \\ du=dx, & v=\frac{1}{3}\sin 3x \end{matrix}\right\}$$

$$= \frac{2}{3}\left(\left[\frac{1}{3}x\sin 3x\right]\Big|_0^{\pi/2} - \frac{1}{3}\int_0^{\pi/2}\sin 3x\,dx\right) =$$

$$= \frac{2}{3}\left(-\frac{\pi}{6} - 0 - \frac{1}{3}\left[-\frac{1}{3}\cos 3x\right]\Big|_0^{\pi/2}\right) = -\frac{\pi}{9} + \frac{2}{27}\cdot(0-1)$$

$$= -\frac{1}{27}(3\pi+2)$$

7. $I=\int_{-5}^{5}\underbrace{x^4\sin 5x}_{f(x)} = \left\{\begin{matrix} f(-x)=(-x)^4\sin(5(-x)) \\ =-x^4\sin 5x=-f(x) \\ f - \text{нечётн} \end{matrix}\right\} = 0.$

8. а) $I_n = \int_1^e \ln^n x\,dx$ — рекурентна формула?

б) $I_4 = \int_1^e \ln^4 x\,dx$

$$I_n = \int_1^e \ln^n x\,dx = \left\{ \begin{array}{ll} u = \ln^n x, & dv = dx \\ du = n\ln^{n-1}x \cdot \frac{1}{x}dx, & v = x \end{array} \right\} =$$

$$= \underbrace{[x\ln^n x]\Big|_1^e}_{=e} - n\underbrace{\int \ln^{n-1}x\,dx}_{=I_{n-1}} = e - n\,I_{n-1}$$

$$I_1 = \int_1^e \ln x\,dx = \left\{ \begin{array}{ll} u = \ln x, & dv = dx \\ du = \frac{dx}{x}, & v = x \end{array} \right\} = [x\ln x]\Big|_1^e - \int_1^e dx$$

$$= e - 0 - x\Big|_1^e = e - (e-1) = 1$$

$I_n = e - nI_{n-1}$, $I_1 = 1$ — рек. ф-ула.

$$I_4 = e - 4I_3 = e - 4(e - 3I_2) = e - 4(e - 3(e - I_1)) =$$
$$= e - 4(e - 3(e-1)) = e - 4(3 - 2e) = e - 12 + 8e = 3(3e - 4)$$

9. а) $I_n = \int_0^{\pi/2} \sin^n x\,dx$ — рекурентна формула

б) $I_{12} = \int_0^{\pi/2} \sin^{12} x\,dx$

$$I_n = \int_0^{\pi/2} \sin^n x\,dx = \int_0^{\pi/2} \sin^{n-1}x \sin x\,dx = \left\{ \begin{array}{ll} u = \sin^{n-1}x, & dv = \sin x\,dx \\ du = (n-1)\sin^{n-2}x\cos x\,dx & \\ v = -\cos x & \end{array} \right\}$$

$$= \underbrace{[-\sin^{n-1}x\cos x]\Big|_0^{\pi/2}}_{= -1\cdot 0 + 0\cdot 1 = 0} + (n-1)\int_0^{\pi/2} \sin^{n-2}x \underbrace{\cos^2 x}_{1-\sin^2 x}\,dx$$

$$= (n-1)\Big(\underbrace{\int_0^{\pi/2} \sin^{n-2}x\,dx}_{=I_{n-2}} - \underbrace{\int_0^{\pi/2} \sin^{n}x\,dx}_{=I_n}\Big)$$

$$n I_n = (n-1) I_{n-2}$$

$$I_n = \frac{n-1}{n} I_{n-2} = \frac{n-1}{n}\,\frac{n-3}{n-2} I_{n-4} = \frac{n-1}{n}\,\frac{n-3}{n-2}\,\frac{n-5}{n-4} I_{n-6} = \dots$$

$$\dots = \begin{cases} \frac{n-1}{n}\,\frac{n-3}{n-2}\,\frac{n-5}{n-4} \cdots \frac{2}{3} I_1, & n \text{ непарно} \\ \frac{n-1}{n}\,\frac{n-3}{n-2}\,\frac{n-5}{n-4} \cdots \frac{3}{4} I_2, & n \text{ парно} \end{cases} =$$

$$I_1 = \int_0^{\pi/2} \sin x\,dx = [-\cos x]\Big|_0^{\pi/2} = 0-(-1) = 1$$

$$I_2 = \int_0^{\pi/2} \sin^2 x\,dx = \int_0^{\pi/2} \frac{1-\cos 2x}{2}\,dx = \frac{1}{2}\Big[x - \frac{1}{2}\sin 2x\Big]\Big|_0^{\pi/2}$$

$$= \frac{1}{2}\Big(\frac{\pi}{2} - 0 - \frac{1}{2}(0-0)\Big) = \frac{\pi}{4} \quad \Big(= \frac{1}{2}\cdot\frac{\pi}{2}\Big)$$

$$= \begin{cases} \frac{n-1}{n}\,\frac{n-3}{n-2}\,\frac{n-5}{n-4} \cdots \frac{2}{3}\cdot 1, & n \text{ непарно} \\ \frac{n-1}{n}\,\frac{n-3}{n-2}\,\frac{n-5}{n-4} \cdots \frac{3}{4}\cdot\frac{1}{2}\,\frac{\pi}{2}, & n \text{ парно} \end{cases} =$$

$$= \begin{cases} \frac{(n-1)!!}{n!!}, & n \text{ непарно} \\ \frac{(n-1)!!}{n!!}\,\frac{\pi}{2}, & n \text{ парно} \end{cases}$$

$$\underset{\text{парно}}{I_{12}} = \frac{(12-1)!!}{12!!}\cdot\frac{\pi}{2} = \frac{11!!}{12!!}\cdot\frac{\pi}{2} = \frac{11\cdot\overset{3}{\cancel{9}}\cdot 7\cdot\cancel{5}\cdot\cancel{3}}{\underset{4}{\cancel{12}}\cdot\underset{2}{\cancel{10}}\cdot 8\cdot\underset{2}{\cancel{6}}\cdot 4\cdot 2}\cdot\frac{\pi}{2} = \frac{231\pi}{2048}$$

10. $I=\int_0^1 \frac{x^2\,dx}{\sqrt{x^2-x+1}}$

Гаус-Остроградски

$$\int \frac{x^2\,dx}{\sqrt{x^2-x+1}} = (Ax+B)\sqrt{x^2-x+1} + \lambda\int\frac{dx}{\sqrt{x^2-x+1}} \quad \Big/\,'_x$$

$$\frac{x^2}{\sqrt{x^2-x+1}} = A\sqrt{x^2-x+1} + (Ax+B)\frac{2x-1}{2\sqrt{x^2-x+1}} + \lambda\cdot\frac{1}{\sqrt{x^2-x+1}} \quad \Big/\cdot 2\sqrt{x^2-x+1}$$

$$2x^2 = 2A + (Ax+B)(2x-1) + 2\lambda$$

$$2x^2 = 2A + 2Ax^2 - Ax + 2Bx - B + 2\lambda$$

$x^2: \quad 2 = 2A \Rightarrow A = 1$

$x: \quad 0 = -A + 2B \Rightarrow B = 1/2$

$1: \quad 0 = 2A - B + \lambda \Rightarrow \lambda = -3/2$

$$\int \frac{x^2\,dx}{\sqrt{x^2-x+1}} = \left(x+\frac{1}{2}\right)\sqrt{x^2-x+1} - \frac{3}{2}\underbrace{\int\frac{dx}{\sqrt{x^2-x+1}}}_{I_1}$$

$$x^2-x+1 = \left(x-\frac{1}{2}\right)^2 + \frac{3}{4} = \frac{3}{4}\left(\left(\frac{\cancel{2}}{\sqrt{3}}\,\frac{2x-1}{\cancel{2}}\right)^2+1\right)$$

$$I_1 = \int\frac{dx}{\sqrt{\frac{3}{4}\left(\left(\frac{2x-1}{\sqrt{3}}\right)^2+1\right)}} = \left|\begin{array}{l}\frac{2x-1}{\sqrt{3}} = sh\,t \\ \frac{2\,dx}{\sqrt{3}} = ch\,t\,dt\end{array}\right| =$$

$$= \int\frac{\frac{\sqrt{3}}{2}\cancel{ch\,t}\,dt}{\frac{\sqrt{3}}{2}\cancel{\sqrt{sh^2t+1}}} = t + C = ash\,\frac{2x-1}{\sqrt{3}} + C$$

$$= \ln\left(\frac{2x-1}{\sqrt{3}} + \sqrt{\left(\frac{2x-1}{\sqrt{3}}\right)^2+1}\right) + C$$

$$\int \frac{x^2\,dx}{\sqrt{x^2-x+1}} = \left(x+\frac{1}{2}\right)\sqrt{x^2-x+1} - \frac{3}{2}\ln\left(\frac{2x-1}{\sqrt{3}} + \sqrt{\left(\frac{2x-1}{\sqrt{3}}\right)^2+1}\right) + C$$

$$\Rightarrow \int_0^1 \frac{x^2\,dx}{\sqrt{x^2-x+1}} = \underline{\frac{3}{2}}\cdot 1 - \frac{3}{2}\ln\left(\frac{1}{\sqrt{3}} + \sqrt{\frac{1}{3}+1}\right) - \underline{\frac{1}{2}}\cdot 1 + \frac{3}{2}\ln\left(-\frac{1}{\sqrt{3}} + \sqrt{\frac{1}{3}+1}\right)$$

$$= \cancel{\frac{3}{2}}\; 1 + \frac{3}{2}\ln\frac{-\frac{1}{\cancel{\sqrt{3}}} + \frac{2}{\cancel{\sqrt{3}}}}{\frac{1}{\cancel{\sqrt{3}}} + \frac{2}{\cancel{\sqrt{3}}}} = 1 + \frac{3}{2}\ln\frac{1}{3}$$

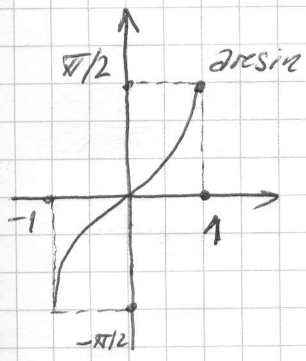
$$= 1 - \frac{3}{2}\ln 3$$

11. $I = \int_{-b}^{b} \overbrace{\sqrt{a^2-x^2}}^{f(x)}\,dx, \quad -a<|b|<a, \quad a,b>0$

$$I = \left\{ \begin{array}{l} f(-x) = \sqrt{a^2-(-x)^2} = \sqrt{a^2-x^2} = f(x) \\ f - \text{парна} \end{array} \right\} = 2\int_0^b \sqrt{a^2-x^2}\,dx$$

$$= \left\{ \begin{array}{l} x = a\sin t, \quad dx = a\cos t\,dt \\ x=0 \Rightarrow t=0, \; x=b, \; t=\arcsin\frac{b}{a} \end{array} \right\} =$$

$$= 2\int_0^{\arcsin\frac{b}{a}} \underbrace{\sqrt{a^2-a^2\sin^2 t}}_{\sqrt{a^2(1-\sin^2 t)} = a\sqrt{\cos^2 t} = a|\cos t|} \cdot a\cos t\,dt = 2a^2\int_0^{\arcsin\frac{b}{a}} |\cos t|\cdot\cos t\,dt \circleddash$$



$-a<|b|<a, \quad a,b>0$

$\Rightarrow \frac{b}{a} < 1, \quad 0 < \frac{b}{a} < 1$

$\frac{b}{a} \in (0,1) \Rightarrow \arcsin\frac{b}{a} \in \left(0, \frac{\pi}{2}\right)$

$\Rightarrow \cos\left(\arcsin\frac{b}{a}\right) \in (0,1)$

$$\circleddash\; 2a^2\int_0^{\arcsin\frac{b}{a}} \underbrace{\cos t\cdot\cos t}_{\cos^2 t}\,dt = \cancel{2}a^2\int_0^{\arcsin\frac{b}{a}} \frac{1+\cos 2t}{\cancel{2}}\,dt =$$

$$= a^2\left[t + \frac{1}{2}\sin 2t\right]\Big|_0^{\arcsin\frac{b}{a}} = a^2\left(\arcsin\frac{b}{2} + \frac{1}{2}\sin\left(2\cdot\arcsin\frac{b}{a}\right)\right)$$