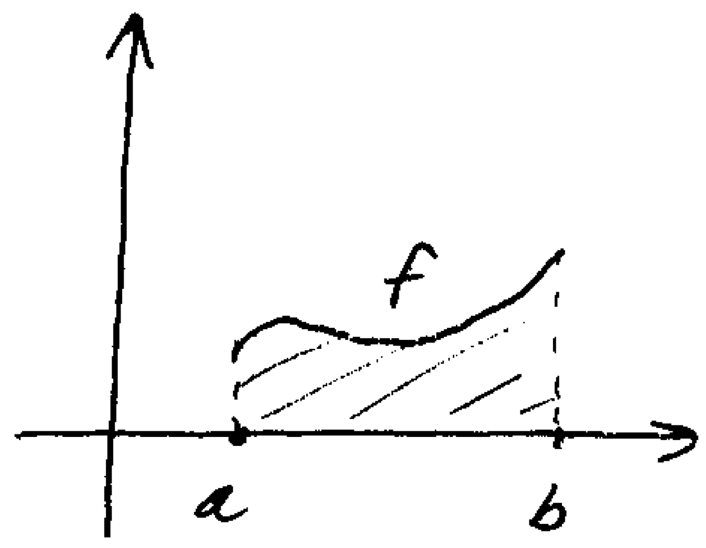
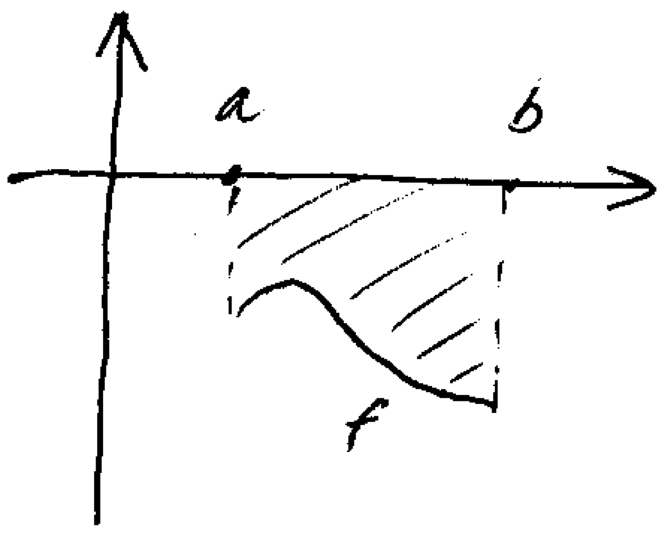
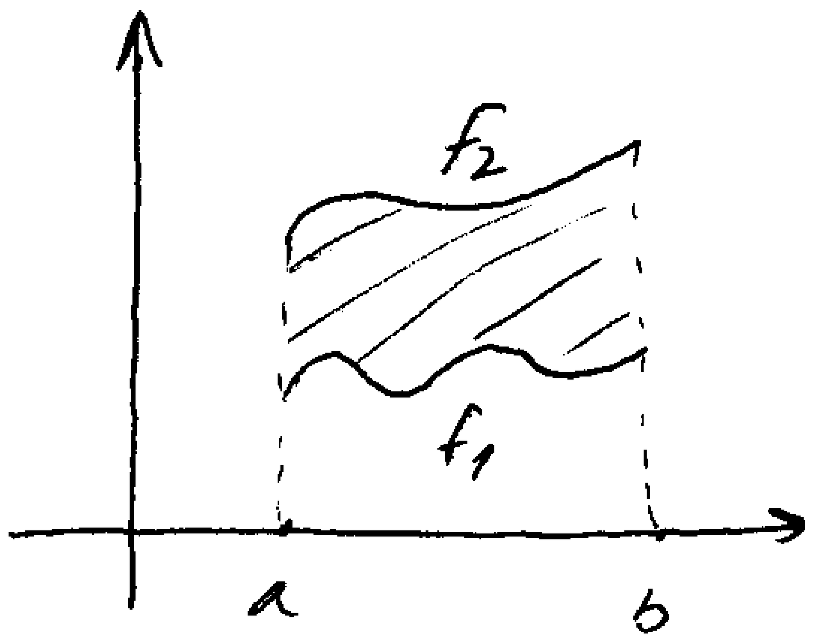
ПРИМЕНЕ ОДРЕЂЕНОГ ИНТЕГРАЛА

I Површина равне фигуре

1) Декартове координате: $y=f(x),\ x\Big|_a^b$







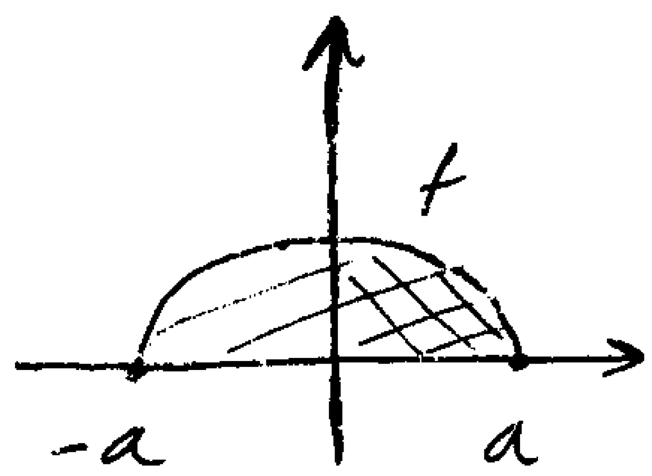
$f(x)\geq 0$

$P=\int_a^b f(x)\,dx$

$f(x)\leq 0$

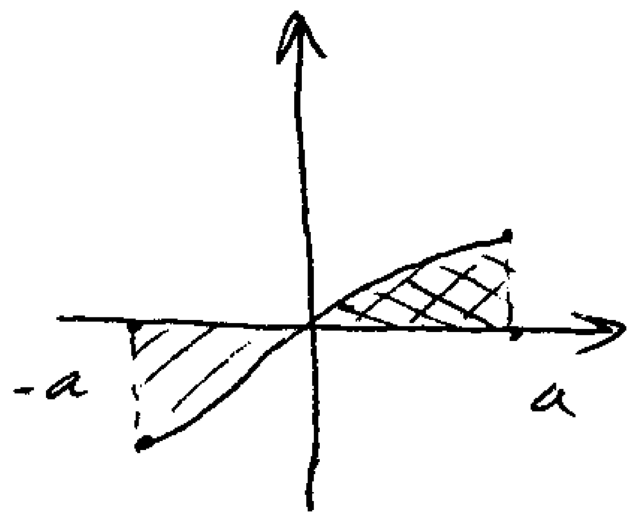
$P=-\int_a^b f(x)\,dx$

$P=\int_a^b \left(f_2(x)-f_1(x)\right)dx$



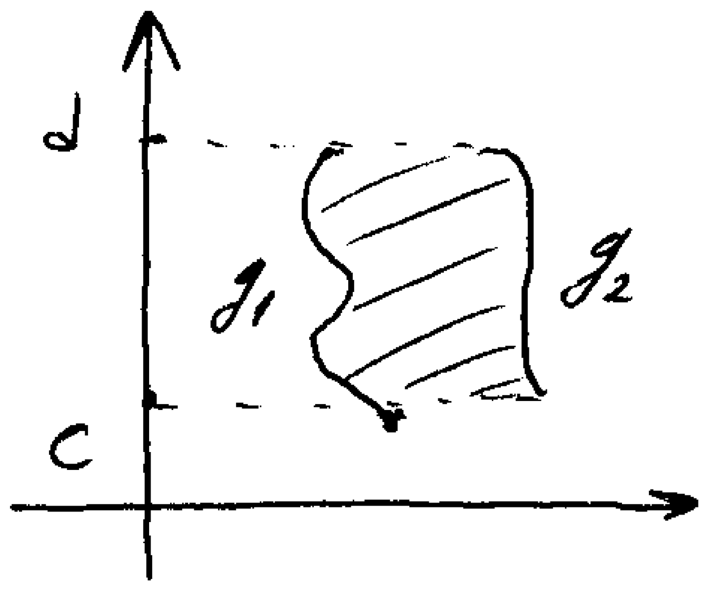
f – парна

$P=2\int_0^a f(x)\,dx$



f – непарна

$P=2\int_0^a f(x)\,dx$



$x=g(y),\ y\Big|_c^d$ (мењамо улоге оса)

$P=\int_c^d \left(g_2(y)-g_1(y)\right)dy$

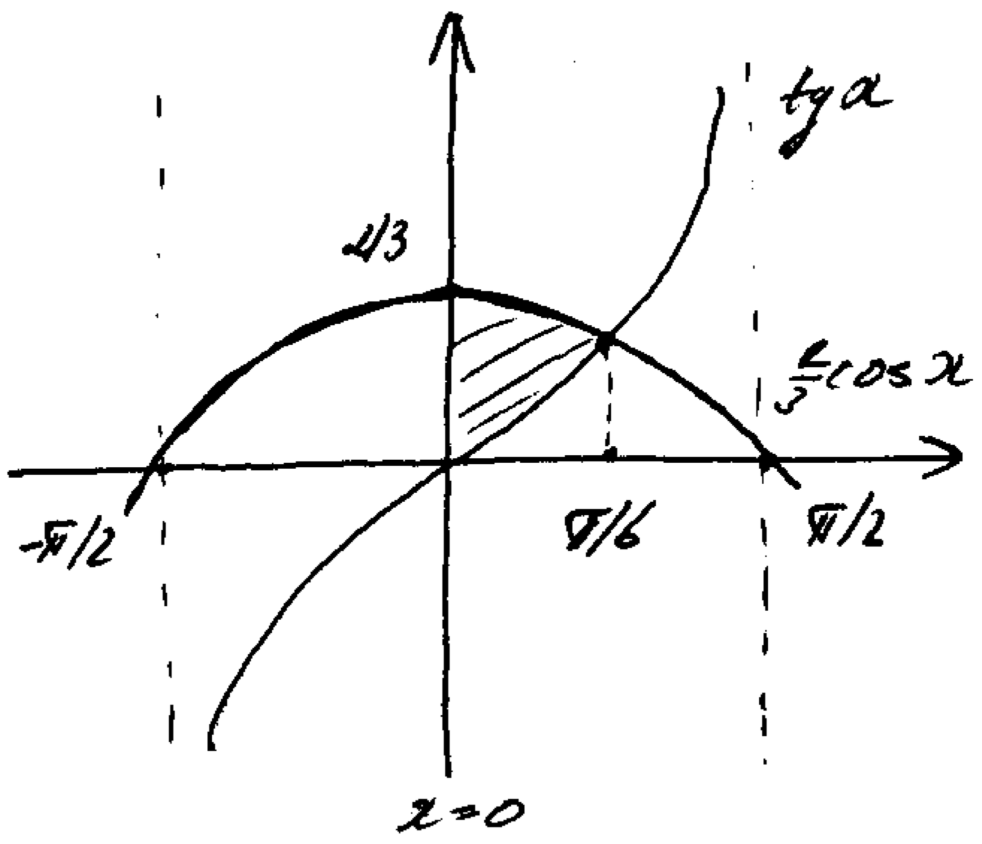
2) Крива задата параметарски: $x=x(t),\ y=y(t),\ t\Big|_\alpha^\beta$

$P=\int_\alpha^\beta y(t)\underbrace{dx(t)}_{x'(t)dt}$

3) Поларне координате: $x=\rho\cos\varphi,\ y=\rho\sin\varphi,\ \rho=\rho(\varphi),\ \varphi\Big|_\alpha^\beta$

$P=\frac{1}{2}\int_\alpha^\beta \rho^2(\varphi)\,d\varphi$

④ Израчунати површину фигуре ограничене линијама $y = tg x$, $y = \frac{2}{3}\cos x$ и $x = 0$.



$$tg x = \frac{2}{3}\cos x$$

$$\frac{\sin x}{\cos x} = \frac{2}{3}\cos x$$

$$3\sin x = 2\cos^2 x = 2(1-\sin^2 x)$$

$$2\sin^2 x + 3\sin x - 2 = 0$$

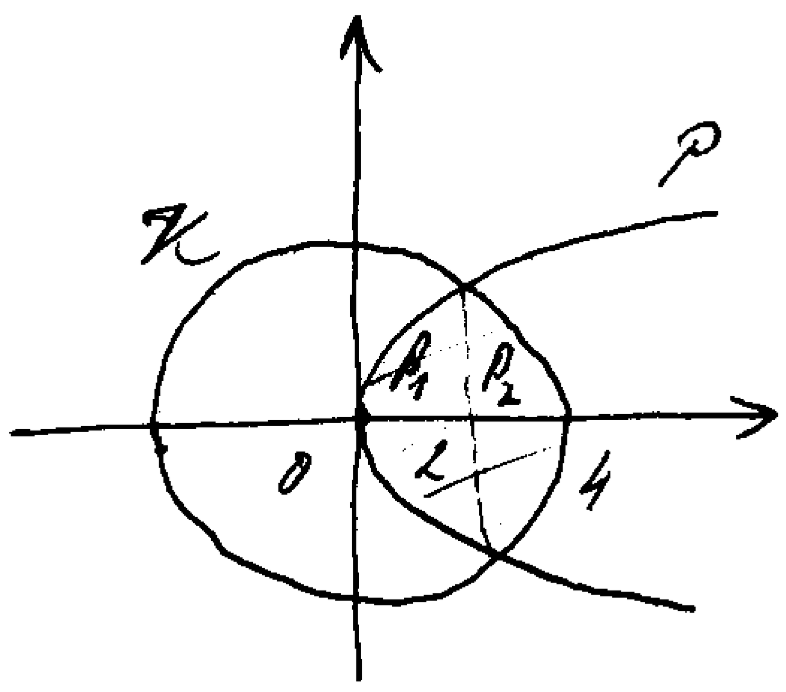
$$(\sin x)_{1/2} = \frac{-3 \pm \sqrt{9+16}}{4} = \frac{-3 \pm 5}{4} \begin{array}{l} \nearrow 1/2 \\ \searrow \cancel{-2} \end{array}$$

$$\sin x = \frac{1}{2}\ ; \quad x = \frac{\pi}{6}\ ; \quad x\Big|_0^{\pi/6}$$

$$P = \int_0^{\pi/6}\left(\frac{2}{3}\cos x - tg x\right)dx = \frac{2}{3}\int_0^{\pi/6}\cos x\,dx - \int_0^{\pi/6}\frac{\sin x}{\cos x}dx =$$

$$= \frac{2}{3}[\sin x]\Big|_0^{\pi/6} + [\ln|\cos x|]\Big|_0^{\pi/6} = \frac{2}{3}\cdot\frac{1}{2} + \ln\frac{\sqrt{3}}{2} = \frac{1}{3} + \ln\frac{\sqrt{3}}{2}$$

(*) Израчунати површину фигуре омеђене кружницом $x^2+y^2=16$ и параболом $y^2=6x$ (изабрати фигуру мање површине)



$$P = 2(P_1+P_2)$$

$$\left.\begin{array}{l} x^2+y^2=16 \\ y^2=6x \end{array}\right\} \quad \begin{array}{l} x^2+6x-16=0 \\ x_1=2,\ \cancel{x_2=-8} \end{array}$$

$$P_1:\ x\Big|_0^2;\quad y^2=6x,\quad y=+\sqrt{6x}$$

$$P_1=\int_0^2\sqrt{6x}\,dx=\sqrt{6}\,\frac{x^{3/2}}{3/2}\Big|_0^2=\sqrt{3}\cdot\sqrt{2}\cdot\frac{2}{3}\cdot 2\sqrt{2}=\frac{8\sqrt{3}}{3}$$

$$P_2:\ x\Big|_2^4;\quad x^2+y^2=16,\quad y=+\sqrt{16-x^2}$$

$$P_2=\int_2^4\sqrt{16-x^2}\,dx=4\int_2^4\sqrt{1-\left(\frac{x}{4}\right)^2}\,dx=\left\{\begin{array}{l}\frac{x}{4}=\sin t\\ \frac{dx}{4}=\cos t\,dt\end{array}\right\}=4\int_{\pi/6}^{\pi/2}\underbrace{\sqrt{1-\sin^2 t}}_{\cos^2 t}\cdot 4\cos t\,dt=$$

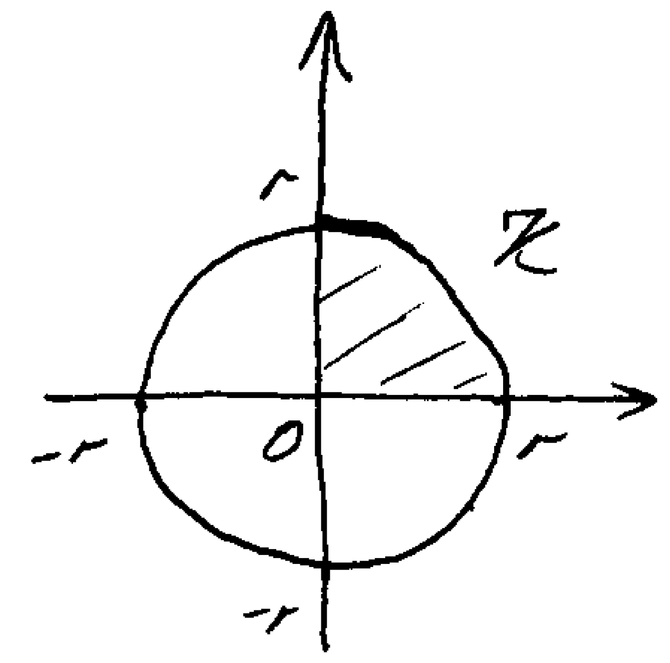
$$=16\int_{\pi/6}^{\pi/2}|\cos t|\cdot\cos t\,dt=\left\{\begin{array}{l}t\Big|_{\pi/6}^{\pi/2},\ \cos t\geq 0\\ |\cos t|=\cos t\end{array}\right\}=16\int_{\pi/6}^{\pi/2}\cos^2 t\,dt=$$

$$=16\int_{\pi/6}^{\pi/2}\frac{1+\cos 2t}{2}\,dt=8\left[t+\frac{1}{2}\sin 2t\right]\Big|_{\pi/6}^{\pi/2}=8\left(\frac{\pi}{2}-\frac{\pi}{6}-\frac{1}{2}\cdot\frac{\sqrt{3}}{2}\right)=$$

$$=\cancel{8}^{\,2}\,\frac{6\pi-2\pi-3\sqrt{3}}{\cancel{12}_{\,3}}=\frac{2}{3}\left(4\pi-3\sqrt{3}\right)$$

$$P=2\left(\frac{8\sqrt{3}}{3}+\frac{8\pi}{3}-\frac{6\sqrt{3}}{3}\right)=\frac{2}{3}\left(8\pi+2\sqrt{3}\right)=\frac{4}{3}\left(4\pi+\sqrt{3}\right)$$

⊛ Применом одређеног интеграла израчунати површину оног дела круга $K(0,r)$ који се налази у првом квадранту.



Познато: $P_K = r^2\pi$

$P = \frac{r^2\pi}{4}$

1. квадрант

$K: x^2+y^2=r^2, \quad y = \pm\sqrt{r^2-x^2}, \quad x\Big|_0^r$

$P = \int_0^r \sqrt{r^2-x^2}\,dx = \left\{ \begin{array}{l} x = r\sin t \\ dx = r\cos t\,dt \end{array} \right\} =$

$= \int_0^{\pi/2} \sqrt{\underline{r^2 - r^2\sin^2 t}} \cdot r\cos t\,dt = r^2 \int_0^{\pi/2} \sqrt{\cos^2 t}\,\cos t\,dt =$

$r^2(1-\sin^2 t) = r^2\cos^2 t$

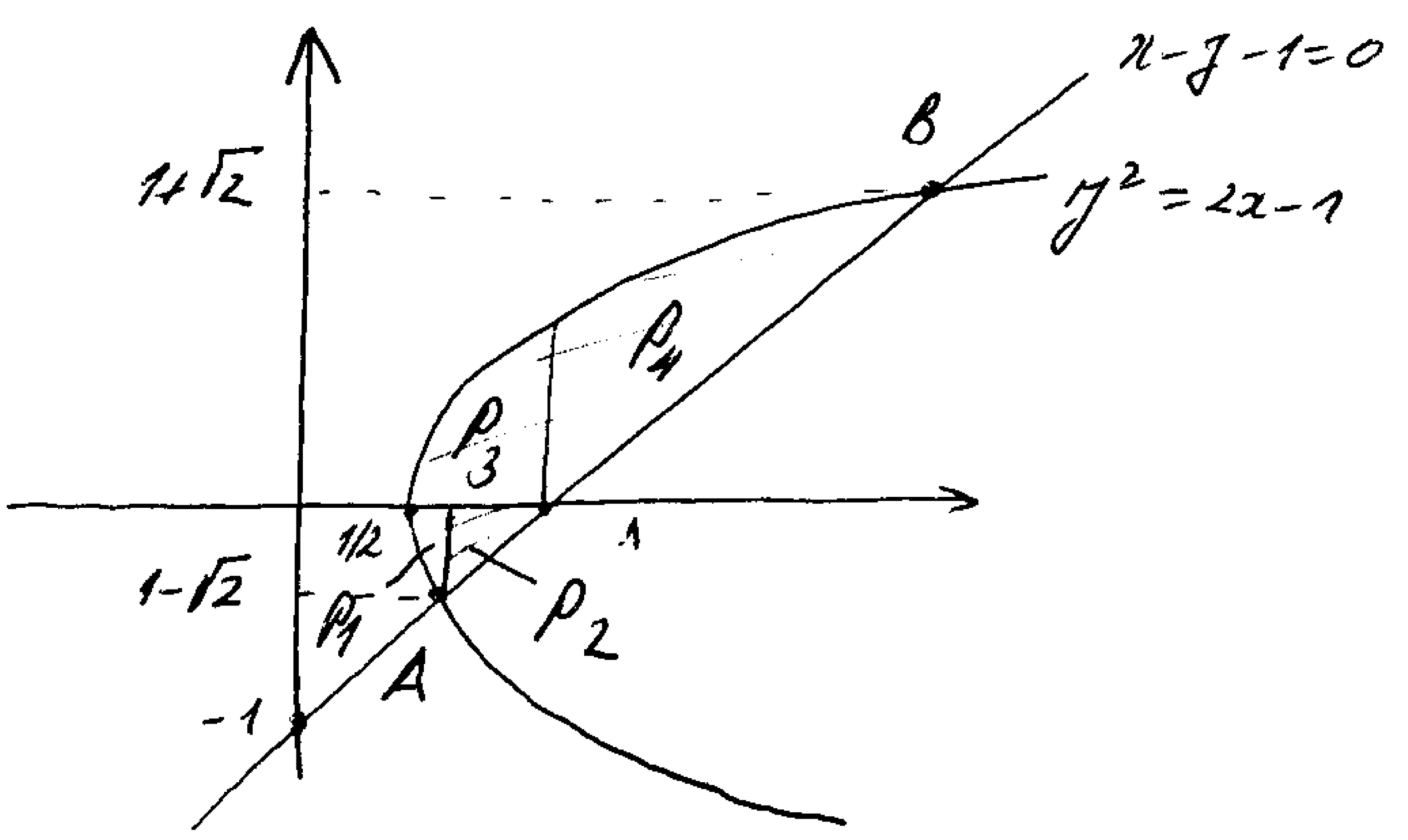
$= r^2 \int_0^{\pi/2} |\cos t| \cdot \cos t\,dt = \left\{ \begin{array}{l} t\Big|_0^{\pi/2}, \ \cos t \geq 0 \\ |\cos t| = \cos t \end{array} \right\} =$

$= r^2 \int_0^{\pi/2} \cos^2 t\,dt = r^2 \int_0^{\pi/2} \frac{1+\cos 2t}{2}\,dt$

$= r^2 \cdot \frac{1}{2}\left[t + \frac{1}{2}\sin 2t\right]\Big|_0^{\pi/2} = \frac{r^2}{2}\cdot\frac{\pi}{2} = \frac{r^2\pi}{4}$

⊛ Израчунати површину фигуре ограничене линијама

$y^2 = 2x-1, \quad x-y-1=0.$



„Стандардно" решавање:

(компликованији начин)

$P = P_1+P_2+P_3+P_4$

$A(x_A, y_A)$, $B(x_B, y_B)$ - тачке пресека праве и параболе

$\left.\begin{array}{l} y^2=2x-1 \\ x-y-1=0 \end{array}\right\} \; x_A = \dots, \; x_B = \dots$

$P_1: \; x\Big|_{1/2}^{x_A}; \quad y = \ominus\sqrt{2x-1}; \quad P_1 = -\int_{1/2}^{x_A} \left(-\sqrt{2x-1}\right) dx = \dots$

$P_2: \; x\Big|_{x_A}^{1}; \quad y = x-1; \quad P_2 = -\int_{x_A}^{1} (x-1)\, dx = \dots$

$P_3: \; x\Big|_{1/2}^{1}; \quad y = \oplus\sqrt{2x-1}; \quad P_3 = \int_{1/2}^{1} \sqrt{2x-1}\, dx = \dots$

$P_4: \; x\Big|_{1}^{x_B}; \quad y_1 = x-1, \; y_2 = \oplus\sqrt{2x-1}; \quad P_4 = \int_1^{x_B} \left(\sqrt{2x-1} - (x-1)\right) dx = \dots$

БРЖЕ: ако заменимо улоге оса

$\left.\begin{array}{l} y^2 = 2x-1 \\ x-y-1=0, \quad x = y+1 \end{array}\right\} \begin{array}{l} y^2-2y-1=0 \\ y_{1/2} = \dfrac{2\pm\sqrt{4+4}}{2} = 1\pm\sqrt2 \end{array} \qquad y\Big|_{1-\sqrt2}^{1+\sqrt2}$

$x_1 = \dfrac{y^2+1}{2}, \quad x_2 = y+1$

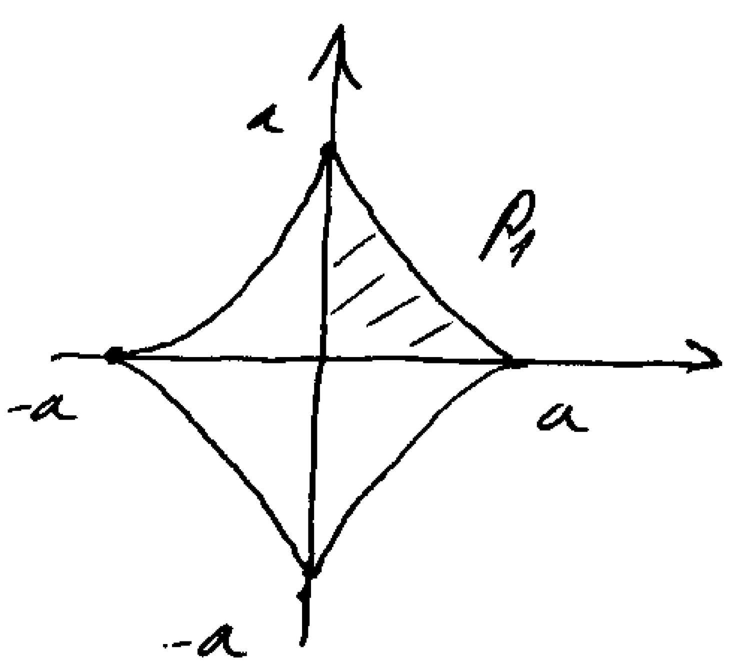
$$P = \int_{1-\sqrt2}^{1+\sqrt2} \left(y+1-\frac{y^2+1}{2}\right) dy = \int_{1-\sqrt2}^{1+\sqrt2} \left(-\frac12 y^2 + y + \frac12\right) dy = \left[-\frac12\frac{y^3}{3} + \frac{y^2}{2} + \frac12 y\right]\Bigg|_{1-\sqrt2}^{1+\sqrt2} =$$

$$= -\frac16\left((1+\sqrt2)^3 - (1-\sqrt2)^3\right) + \frac12\left((1+\sqrt2)^2 - (1-\sqrt2)^2\right) + \frac12\left((1+\sqrt2)-(1-\sqrt2)\right) =$$

$$= -\frac16\left(\cancel{1}+3\sqrt2+\cancel{6}+2\sqrt2-\cancel{1}+3\sqrt2-\cancel{6}+2\sqrt2\right) + \frac12\left(\cancel{1}+2\sqrt2+\cancel{2}-\cancel{1}+2\sqrt2-\cancel{2}\right) + \sqrt2 = \left(-\frac53+2+1\right)\sqrt2 = \frac{4\sqrt2}{3}$$

(*) Израчунати површину фигуре омеђене кривом

$\underbrace{x^{2/3}+y^{2/3}=a^{2/3}}_{\text{АСТРОИДА}}$ $(a>0)$.



$P=4P_1$

„Стандардно" решавање:

$x\Big|_0^a$

$y^{2/3}=a^{2/3}-x^{2/3}$

$y=\pm\sqrt{(a^{2/3}-x^{2/3})^3}$

$P=4\int_0^a\sqrt{(a^{2/3}-x^{2/3})^3}\,dx=\ldots$ компликовано

параметризација:

$x^{2/3}+y^{2/3}=a^{2/3} \quad /:a^{2/3}$

$\left(\sqrt[3]{\frac{x}{a}}\right)^2+\left(\sqrt[3]{\frac{y}{a}}\right)^2=1$

$\sqrt[3]{\frac{x}{a}}=\cos t, \quad \sqrt[3]{\frac{y}{a}}=\sin t$

$x=a\cos^3 t, \quad y=a\sin^3 t$

$dx=a\cdot 3\cos^2 t(-\sin t)\,dt$

$x=0: \quad a\cos^3 t=0, \ \cos t=0, \ t=\pi/2$

$x=a: \quad a\cos^3 t=a, \ \cos t=1, \ t=0 \qquad t\Big|_{\pi/2}^0$

$P=4P_1=4\int_{\pi/2}^0 a\sin^3 t(-3a\sin t\cos^2 t)\,dt=12a^2\int_0^{\pi/2}\sin^4 t\underbrace{\cos^2 t}_{1-\sin^2 t}\,dt=$

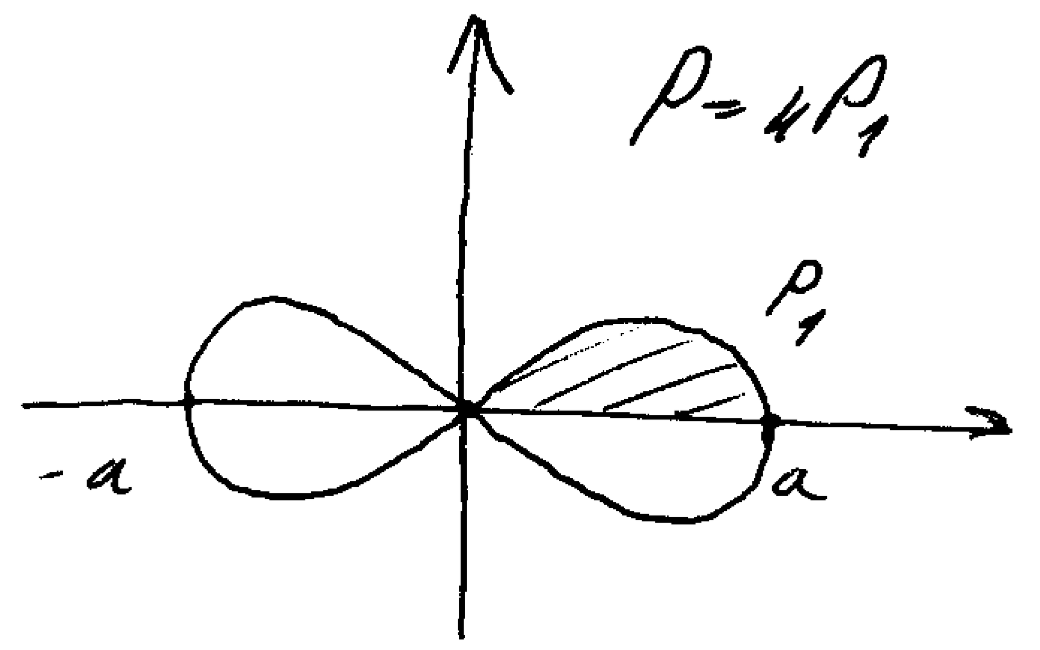
$\left(P_1=\int_\alpha^\beta y(t)x'(t)\,dt\right)$

$=12a^2\left(\int_0^{\pi/2}\sin^4 t\,dt-\int_0^{\pi/2}\sin^6 t\,dt\right)=12a^2\left(\frac{3!!}{4!!}\cdot\frac{\pi}{2}-\frac{5!!}{6!!}\cdot\frac{\pi}{2}\right)=$

$=\cancel{12}^{\,3}a^2\pi\left(\frac{3}{4\cdot\cancel{2}}-\frac{5\cdot\cancel{3}}{\cancel{6}_{\,2}\cdot 4\cdot 2}\right)=3a^2\pi\,\frac{6-5}{8}=\frac{3a^2\pi}{8}$

(*) Израчунати површину фигуре омеђене кривом

$$\underbrace{(x^2+y^2)^2 = a^2(x^2-y^2)}_{\text{ЛЕМНИСКАТА}} \quad (a>0).$$



Поларне координате: $x = \rho\cos\varphi,\ y = \rho\sin\varphi$

$$(x^2+y^2)^2 = a^2(x^2-y^2)$$

$$(\rho^2\cos^2\varphi + \rho^2\sin^2\varphi)^2 = a^2(\rho^2\cos^2\varphi - \rho^2\sin^2\varphi)$$

$$\rho^4 = a^2\rho^2(\cos^2\varphi - \sin^2\varphi)$$

$$\rho^2 = a^2\cos 2\varphi \quad (\rho = a\sqrt{\cos 2\varphi})$$

$\downarrow$

$$\left.\begin{array}{l} \cos 2\varphi \geq 0,\quad 2\varphi\Big|_{-\pi/2}^{\pi/2},\quad \varphi\Big|_{-\pi/4}^{\pi/4} \\ P_1 - 1.\ \text{квадрант}:\ \varphi\Big|_0^{\pi/2} \end{array}\right\}\ \varphi\Big|_0^{\pi/4}$$

$$P = 4P_1 = 4\frac{1}{2}\int_\alpha^\beta \rho^2(\varphi)\,d\varphi = 2\int_0^{\pi/4} a^2\cos 2\varphi\,d\varphi = 2a^2\left[\frac{1}{2}\sin 2\varphi\right]\Big|_0^{\pi/4} = a^2$$