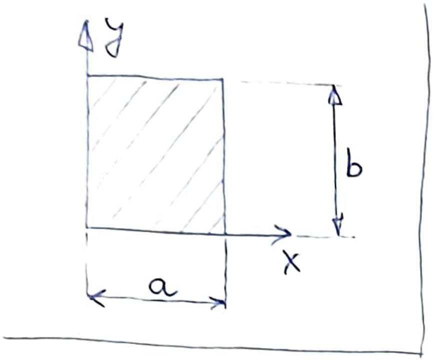
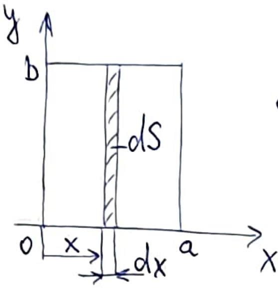
1. Одредити положај тежишта правоугаоне плоче дате на слици.




Решење:

$$\vec{r}_c = \frac{\int_S \vec{r}\,dS}{S}, \quad S = ab$$

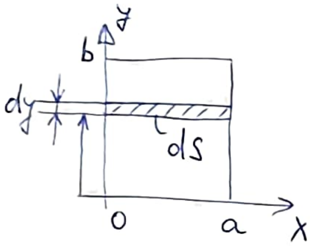
$$x_c = \frac{\int_S x\,dS}{S}; \quad y_c = \frac{\int_S y\,dS}{S}$$




$$dS = b\,dx$$

$$x_c = \frac{\int x\,b\,dx}{ab} = \frac{b}{ab}\int_0^a x\,dx$$

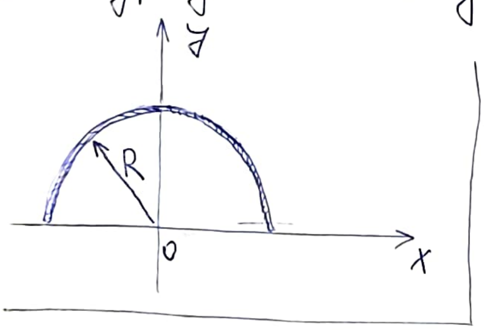
$$x_c = \frac{1}{a}\,\frac{x^2}{2}\Big|_0^a, \quad \boxed{x_c = \frac{a}{2}}$$



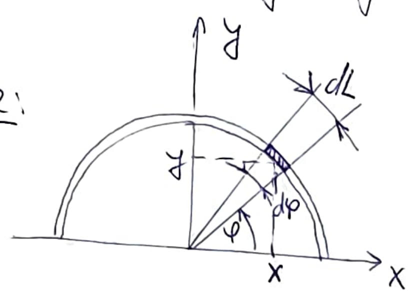

$$dS = a\,dy$$

$$y_c = \frac{\int y\,a\,dy}{ab} = \frac{a}{ab}\int_0^b y\,dy = \frac{1}{b}\,\frac{y^2}{2}\Big|_0^b, \quad \boxed{y_c = \frac{b}{2}}$$

2. Одредити положај тежишта кружног лука датог на слици.




Решење:




$$\vec{r}_c = \frac{\int_L \vec{r}\,dL}{L}$$

$$x_c = \frac{\int_L x\,dL}{L}$$

$$y_c = \frac{\int_L y\,dL}{L}$$

$$x = R\cos\varphi$$

$$y = R\sin\varphi$$

$$dL_\varphi = R\,d\varphi, \quad L = R\pi$$

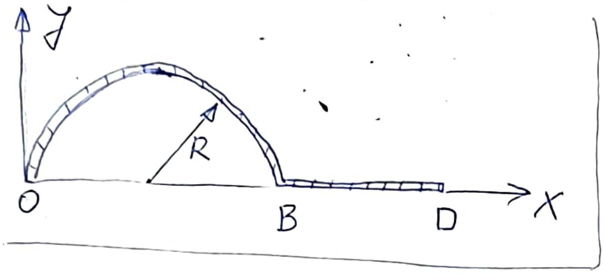
$$x_c = \frac{\int x\,dL}{L} = \frac{\int_0^\pi R\cos\varphi \cdot R\,d\varphi}{R\pi}$$

$$x_c = \frac{R^2}{R\pi}\int_0^\pi \cos\varphi\,d\varphi = \frac{R}{\pi}\sin\varphi\Big|_0^\pi = \frac{R}{\pi}(0-0) = 0$$

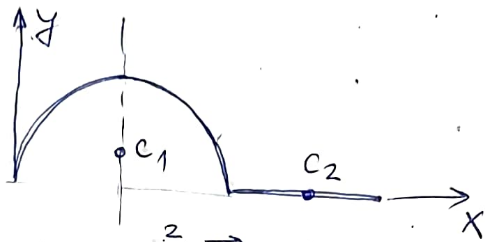
$$y_c = \frac{\int y\,dL}{L} = \frac{\int_0^\pi R\sin\varphi\,R\,d\varphi}{R\pi} = \frac{R^2}{R\pi}\int_0^\pi \sin\varphi\,d\varphi = \frac{R}{\pi}(-\cos\varphi)\Big|_0^\pi$$

$$y_c = -\frac{R}{\pi}(\cos\pi - \cos 0), \quad y_c = \frac{2R}{\pi} \Rightarrow \boxed{C\left(0, \frac{2R}{\pi}\right)}$$

3) Одредити положај тежишта линијске контуре приказане на слици ($\overline{BD} = R$).




Решење:




$$\vec{r}_c = \frac{\sum_{i=1}^{2} \vec{r}_{ci}\,\Delta L_i}{\sum_{i=1}^{2} \Delta L_i} = \frac{\vec{r}_{c1}\,\Delta L_1 + \vec{r}_{c2}\,\Delta L_2}{\Delta L_1 + \Delta L_2}$$

$$x_c = \frac{x_{c1}\Delta L_1 + x_{c2}\Delta L_2}{\Delta L_1 + \Delta L_2}, \quad y_c = \frac{y_{c1}\Delta L_1 + y_{c2}\Delta L_2}{\Delta L_1 + \Delta L_2}$$

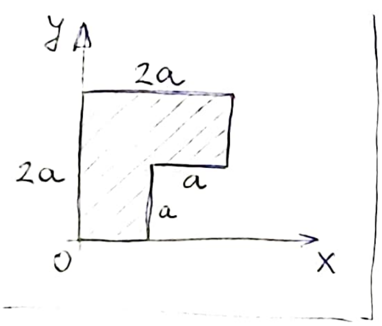
$$x_{c1} = R, \quad y_{c1} = \frac{2R}{\pi}, \quad \Delta L_1 = R\pi; \quad x_{c2} = 2R + \frac{R}{2} = \frac{5}{2}R$$

$$y_{c2} = 0, \quad \Delta L_2 = R$$

$$x_c = \frac{R \cdot R\pi + \frac{5}{2}R \cdot R}{R\pi + R} = \frac{R^2}{R(\pi+1)}\left(\pi + \frac{5}{2}\right) = \boxed{\frac{R}{\pi+1}\left(\pi + 5/2\right)}$$

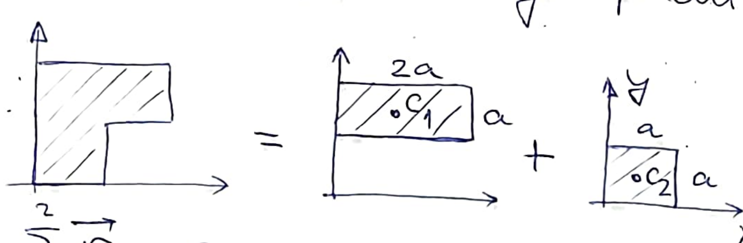
$$y_c = \frac{\frac{2R}{\cancel{\pi}}\, R\cancel{\pi} + 0 \cdot R}{R\pi + R} = \frac{2R^2}{R(\pi+1)} = \boxed{\frac{2R}{\pi+1}}$$

4) Одредити координате плоче приказане на слици.




Решење!

I Метода раставља




$$\vec{r}_c = \frac{\sum_{i=1}^{2} \vec{r}_{ci}\,\Delta S_i}{S} = \frac{\vec{r}_{c1}\,\Delta S_1 + \vec{r}_{c2}\,\Delta S_2}{\Delta S_1 + \Delta S_2}$$

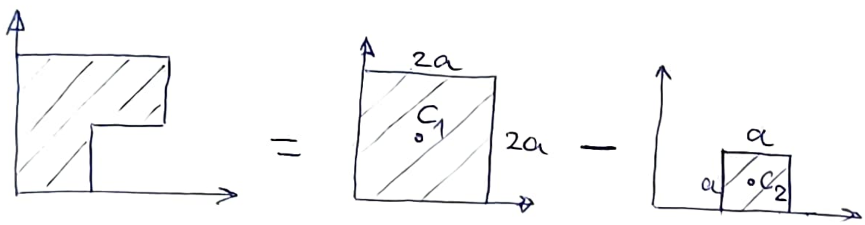
$$x_c = \frac{x_{c1}\Delta S_1 + x_{c2}\Delta S_2}{S} = \frac{a(2a \cdot a) + \frac{a}{2}(a \cdot a)}{2a^2 + a^2} = \frac{a^3(2 + 1/2)}{3a^2}$$

$$X_c = \frac{\frac{5}{2}a}{3} = \boxed{\frac{5}{6}a}$$

$$Y_c = \frac{Y_{c1}\Delta S_1 + Y_{c2}\Delta S_2}{S} = \frac{\frac{3}{2}a\cdot 2a^2 + \frac{a}{2}a^2}{3a^2} = \frac{(3+1/2)a^3}{3a^2}$$

$$\boxed{Y_c = \frac{7}{6}a}$$

## II Метода „негативних тежина"




$$\vec{r}_c = \frac{\vec{r}_{c1}\Delta S_1 - \vec{r}_{c2}\Delta S_2}{\Delta S_1 - \Delta S_2}$$

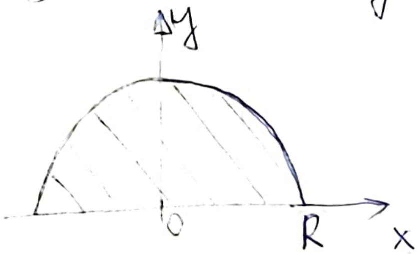
$$X_c = \frac{X_{c1}\Delta S_2 - X_{c2}\Delta S_2}{\Delta S_1 - \Delta S_2} = \frac{a\cdot(2a\cdot 2a) - \frac{3}{2}a(a\cdot a)}{4a^2 - a^2} = \frac{4a^3 - \frac{3}{2}a^3}{3a^2}$$

$$X_c = \frac{5/2\, a^3}{3a^2} = \boxed{\frac{5}{6}a}$$

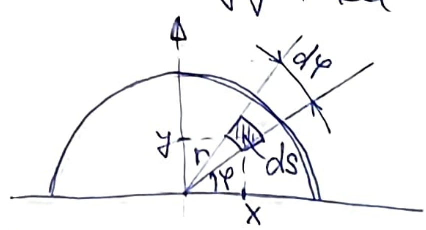
$$Y_c = \frac{Y_{c1}\Delta S_1 - Y_{c2}\Delta S_2}{\Delta S_1 - \Delta S_2} = \frac{a\cdot 4a^2 - \frac{a}{2}a^2}{3a^2} = \frac{(4-1/2)a^3}{3a^2} = \frac{7/2\, a}{3}$$

$$\boxed{Y_c = \frac{7}{6}a}$$

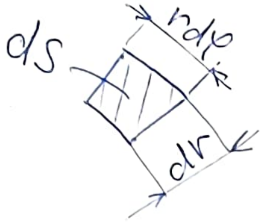
5. Одредити положај тежишта полудиска приказаног на слици.




Решење:




$$\vec{r}_c = \frac{\int_S \vec{r}\,ds}{S}\ ; \quad X_c = \frac{\int_S x\,ds}{S}\ , \quad Y_c = \frac{\int_S y\,ds}{S}$$




$$ds = dr \cdot r\,d\varphi$$

$$x = r\cos\varphi \; ; \; y = r\sin\varphi \; , \quad S = \frac{R^2\pi}{2}$$

$$x_c = \frac{\iint_S x\,ds}{S} = \frac{\int_0^{\pi}\int_0^{R} r\cos\varphi \cdot dr \cdot r\,d\varphi}{R^2\pi/2} = \frac{1}{R^2\pi/2}\int_0^R r^2\,dr \cdot \int_0^{\pi}\cos\varphi\,d\varphi$$

$$x_c = \frac{2}{R^2\pi}\,\frac{r^3}{3}\Big|_0^R \cdot \sin\varphi\Big|_0^{\pi} = \frac{2}{3R^2\pi}(R^3-0)\cdot(\underbrace{\sin\pi}_{0}-\underbrace{\sin 0}_{0}) = 0$$

$$y_c = \frac{\iint_S y\,ds}{S} = \frac{\int_0^{\pi}\int_0^{R} r\sin\varphi\,dr \cdot r\,d\varphi}{R^2\pi/2} = \frac{2}{R^2\pi}\int_0^R r^2\,dr\int_0^{\pi}\sin\varphi\,d\varphi$$

$$y_c = \frac{2}{R^2\pi}\,\frac{r^3}{3}\Big|_0^R \cdot (-\cos\varphi)\Big|_0^{\pi} = -\frac{2}{3R^2\pi}(R^3-0)(\cos\pi-\cos 0)$$

$$y_c = -\frac{2R}{3\pi}(-1-1)\,, \quad \boxed{y_c = \frac{4R}{3\pi}}\,, \Rightarrow C\left(0, \frac{4R}{3\pi}\right)$$