1. (II kolokvijum 25. maj 2024.)

Reši diferencijalnu jednačinu $y' + \frac{2y}{x} = x^2 y^2 \sin x$

J-nu ćemo napisati u obliku: $y' + \underbrace{\frac{2}{x}}_{P(x)} y = \underbrace{x^2 \sin x}_{Q(x)} y^2$ (Bernulijeva) $\alpha = 2$

Smena: $z = y^{1-\alpha}$, $z = z(x)$

$z = y^{-1}$

JNA se svodi na $\frac{z'}{1-\alpha} + P(x) z = Q(x)$ (linearna)

$$\frac{z'}{-1} + \frac{2}{x} z = x^2 \sin x$$

$$z' \underbrace{- \frac{2}{x}}_{p(x)} z = \underbrace{-x^2 \sin x}_{q(x)}$$

pa je rešenje linearne:

$$z = e^{-\int p(x)dx}\left(C + \int q(x) e^{\int p(x)dx} dx\right)$$

$$\int p(x)dx = \int -\frac{2}{x} dx = -2\ln|x|$$

$$e^{2\ln|x|} = x^2$$

$$z = e^{-\int -\frac{2}{x}dx}\left(C + \int -x^2 \sin x \cdot e^{\int -\frac{2}{x}dx} dx\right) =$$

$$= e^{2\ln|x|}\left(C - \int x^2 \sin x \, e^{-2\ln|x|} dx\right) =$$

$$= x^2 \cdot \left(C - \int x^2 \sin x \cdot \frac{1}{x^2} dx\right) =$$

$$= x^2 \cdot \left(C - \int \sin x dx\right) = x^2 (C + \cos x)$$

$$z = \frac{1}{y}$$

$$y = \frac{1}{z}$$

$$\boxed{y = \frac{1}{x^2 (C + \cos x)}}$$

2. (II kolokv. 2023)

Reši diferencijalnu jednačinu $y' = \dfrac{y+2x-6}{2y+x}$.

Jednačina se smenom $\begin{array}{l} x = X+A \\ y = Y+B \end{array}$ svodi na homogenu za neke A i B.

$$Y' = \frac{Y+B+2\cdot(X+A)-6}{2(Y+B)+X+A}$$

$$Y' = \frac{2X+Y+2A+B-6}{X+2Y+A+2B}$$

Biramo A i B tako da važi:

$$2A+B-6=0$$
$$A+2B=0 \; /\cdot(-2)$$
$$-3B=6$$
$$\boxed{B=-2} \quad \boxed{A=4}$$

$x = X+4 \Rightarrow X = x-4$

$y = Y-2 \Rightarrow Y = y+2$

Polazna jednačina postaje

$$Y' = \frac{2X+Y}{X+2Y} \Rightarrow Y' = \frac{2+\frac{Y}{X}}{1+2\frac{Y}{X}} \quad \text{(homogena)}$$

Smenom $z = \dfrac{Y}{X}$ $(Y' = z'X+z)$, dobijamo

$$z'X+z = \frac{2+z}{1+2z}$$

$$z'X = \frac{2+z}{1+2z} - z$$

$$z'X = \frac{2+z-z-2z^2}{1+2z}$$

$$z'X = \frac{-2z^2+2}{1+2z}$$

$$\frac{1+2z}{-2z^2+2}\,dz = \frac{dX}{X} \; /\int$$

$$\frac{1}{-2}\int\frac{1+2z}{z^2-1}\,dz = \int\frac{dX}{X}$$

$$-\frac{1}{2}\int\frac{1}{z^2-1} - \frac{1}{2}\int\frac{2z}{z^2-1}\,dz = \ln|X|$$

$$-\frac{1}{4}\ln\left|\frac{z-1}{z+1}\right| - \frac{1}{2}\ln|z^2-1| + C = \ln|X| \; /\cdot(-4)$$

$$\ln\left|\frac{z-1}{z+1}\right| + 2\ln|z^2-1| - 4C = -4\ln|X|$$

$$\ln|z-1|^3\cdot|z+1| + 4\ln|X| = 4C^{\,C_1}$$

$$\ln|z-1|^3|z+1|\cdot X^4 = C_1$$

$$|z-1|^3|z+1|\cdot X^4 = e^{C_1}{}^{\,C_2}$$

$$\left|\frac{Y}{X}-1\right|^3\cdot\left|\frac{Y}{X}+1\right|\cdot X^4 = C_2$$

$$|Y-X|^3\cdot|Y+X| = C_2$$

$$|y+2-x+4|^3\cdot|y+2+x-4| = C_2$$

$$\boxed{|y-x+6|^3\cdot|y+x-2| = C_2}$$

3. (II kolokv. 2024.)

$$\int_{-\pi/2}^{0} \frac{2-\sin x}{2+\cos x}\,dx = ?$$

smena: $tg\frac{x}{2}=t \Rightarrow \sin x = \frac{2t}{1+t^2}$

$\cos x = \frac{1-t^2}{1+t^2}$

$dx = \frac{2dt}{1+t^2}$

x	$-\pi/2$	0
t	-1	0

$tg\left(-\frac{\pi}{4}\right) = -1$

$tg\frac{0}{2} = tg\,0 = 0$

$$I = \int_{-1}^{0} \frac{2-\frac{2t}{1+t^2}}{2+\frac{1-t^2}{1+t^2}} \cdot \frac{2dt}{1+t^2} = \int_{-1}^{0} \frac{\frac{2+2t^2-2t}{\cancel{1+t^2}}}{\frac{2+2t^2+1-t^2}{\cancel{1+t^2}}} \cdot \frac{2dt}{1+t^2} = 2\int_{-1}^{0} \frac{2+2t^2-2t}{3+t^2} \cdot \frac{dt}{1+t^2} =$$

$$= 4 \cdot \int_{-1}^{0} \frac{(t^2-t+1)\,dt}{(t^2+1)(t^2+3)}$$ ↘ RACIONALNA

$$\frac{t^2-t+1}{(t^2+1)(t^2+3)} = \frac{At+B}{t^2+1} + \frac{Ct+D}{t^2+3} \quad \Big| \cdot (t^2+1)(t^2+1)$$

$$t^2-t+1 = (At+B)(t^2+3) + (Ct+D)\cdot(t^2+1)$$

$$t^2-t+1 = (A+C)t^3 + (B+D)t^2 + (3A+C)t + 3B+D$$

$A+C=0 \;/\cdot(-1)$

$B+D=1 \;/\cdot(-1)$

$3A+C=-1$ +

$3B+D=1$ +

$\boxed{B=0}$

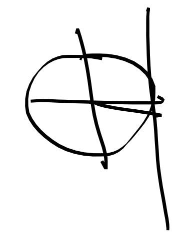
$\boxed{D=1}$

$\boxed{A=-\frac{1}{2}}$

$\boxed{C=\frac{1}{2}}$

$$I = 4\cdot\int_{-1}^{0}\frac{-\frac{1}{2}t}{t^2+1}\,dt + 4\int_{-1}^{0}\frac{\frac{1}{2}t+1}{t^2+3}\,dt$$

$$= -\ln|t^2+1|\Big|_{-1}^{0} + \ln|t^2+3|\Big|_{-1}^{0} + \frac{4}{\sqrt3}\,\mathrm{arctg}\frac{t}{\sqrt3}\Big|_{-1}^{0}$$

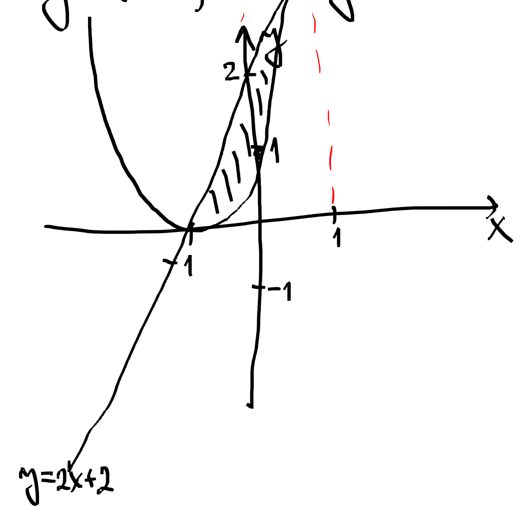


$$= -(0-\ln 2) + (\ln 3 - \ln 4) + \frac{4}{\sqrt3}\left(\mathrm{arctg}\,0 - \mathrm{arctg}\left(-\frac{\sqrt3}{3}\right)\right)$$

$$= \underbrace{\ln 2 + \ln\frac{3}{4}} + \frac{4}{\sqrt3}\left(0-\left(-\frac{\pi}{6}\right)\right)$$

$$= \ln\frac{3}{2} + \frac{\cancel{4}^{2}\pi}{\cancel{6}_{3}\sqrt3} = \boxed{\frac{2\pi\sqrt3}{9} - \ln\frac{3}{2}}$$

4. Naći površinu figure ograničene krivama $y=(1+x)^2$, $y=2x+2$.



Presek: $\left.\begin{array}{l} y=(1+x)^2 \\ y=2x+2 \end{array}\right\}$ sistem

$$(1+x)^2 = 2x+2$$

$$x^2+\cancel{2x}+1 = \cancel{2x}+2$$

$$x^2=1$$

$x=1$ $\quad$ $x=-1$

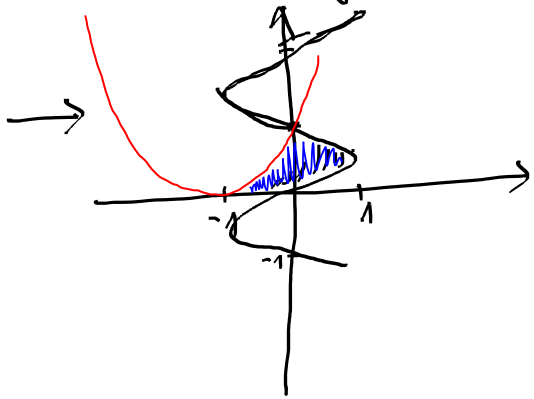
$y=4$ $\quad$ $y=0$

$$P=\int_{-1}^{1}\left(2x+2-(1+x)^2\right)dx = \int_{-1}^{1}\left(2x+2-1-2x-x^2\right)dx = \int_{-1}^{1}(1-x^2)\,dx = \left(x-\frac{x^3}{3}\right)\Big|_{-1}^{1} =$$

$$= \left(1-\frac{1}{3}\right)-\left(-1+\frac{1}{3}\right) = 2-\frac{2}{3} = \boxed{\frac{4}{3}}$$

5. Naći površinu figure ograničene krivama $\sin(\pi y)=x$, $y=(1+x)^2$ i $y=0$ za $0\le y\le 1$.

POSMATRAMO PO y-OSI:



$$P=\int_0^1\left(\sin(\pi y)-(\sqrt{y}-1)\right)dy$$

$$=\int_0^1 \sin(\pi y)\,dy-\int_0^1\sqrt{y}\,dy+\int_0^1 dy$$

$$=-\frac{1}{\pi}\cos(\pi y)\Big|_0^1-\frac{y^{\frac{3}{2}}}{\frac{3}{2}}\Big|_0^1+y\Big|_0^1=$$

$$=-\frac{1}{\pi}(-1-1)-\frac{2}{3}+1=\boxed{\frac{2}{\pi}+\frac{1}{3}}$$

$y=(1+x)^2$

$1+x=\sqrt{y}$ ili $1+x=-\sqrt{y}$

$x=\sqrt{y}-1$ $\quad$ $x=-1-\sqrt{y}$

Pošto je $0\le y\le 1 \Rightarrow x\in[-1,0]$

6. (II kolokv. 2022.)

$$\int_0^{\infty}\left(\sqrt{2}-\sqrt{2+e^{-x}}\right)dx \quad \to \text{nesvojstven}$$

$$I=\lim_{b\to\infty}\int_0^{b}\left(\sqrt{2}-\sqrt{2+e^{-x}}\right)dx=\lim_{b\to\infty}\int_0^{b}\left(\sqrt{2}-\sqrt{2+e^{-x}}\right)\cdot\frac{\sqrt{2}+\sqrt{2+e^{-x}}}{\sqrt{2}+\sqrt{2+e^{-x}}}\,dx=$$

$$=\lim_{b\to\infty}\int_0^{b}\frac{2-2-e^{-x}}{\sqrt{2}+\sqrt{2+e^{-x}}}\,dx=\lim_{b\to\infty}\int_0^{b}\frac{-e^{-x}dx}{\sqrt{2}+\sqrt{2+e^{-x}}}\ * =$$

$$*\ \int\frac{-e^{-x}dx}{\sqrt{2}+\sqrt{2+e^{-x}}}=\begin{bmatrix}2+e^{-x}=t^2\\ -e^{-x}dx=2t\,dt\end{bmatrix}=\int\frac{2t\,dt}{\sqrt{2}+t}=2\int\frac{t\,dt}{t+\sqrt{2}}=2\int\frac{(t+\sqrt{2})-\sqrt{2}}{t+\sqrt{2}}dt=$$

$$=2\int 1\,dt-2\sqrt{2}\int\frac{1}{t+\sqrt{2}}dt=2t-2\sqrt{2}\ln|t+\sqrt{2}|=2\sqrt{2+e^{-x}}-2\sqrt{2}\ln\left|\sqrt{2+e^{-x}}+\sqrt{2}\right|$$

$$I=\lim_{b\to\infty}\left(2\sqrt{2+e^{-x}}-2\sqrt{2}\ln\left|\sqrt{2+e^{-x}}+\sqrt{2}\right|\right)\Big|_0^{b}=$$

$$\lim_{b\to\infty}e^{-b}=0$$

$$=\lim_{b\to\infty}\left(2\sqrt{2+e^{-b}}-2\sqrt{2}\ln\left|\sqrt{2+e^{-b}}+\sqrt{2}\right|-2\sqrt{3}+2\sqrt{2}\ln\left|\sqrt{3}+\sqrt{2}\right|\right)=$$

$$=\boxed{2\sqrt{2}-2\sqrt{2}\ln 2\sqrt{2}-2\sqrt{3}+2\sqrt{2}\ln(\sqrt{3}+\sqrt{2})}$$

7. (Jul 2014.)

Naći dužinu luka krive $y=1-e^{-x}$ za $0\le x\le 2$.

$$L=\int_a^b \sqrt{1+(y'(x))^2}\,dx \qquad y'=e^{-x}$$

$$L=\int_0^2 \sqrt{1+(e^{-x})^2}\,dx=\int_0^2\sqrt{1+e^{-2x}}\,dx$$

$\Big[$ smena $1+e^{-2x}=t^2$

x	0	2
t		

$$-2e^{-2x}dx=2t\,dt$$

$$-e^{-2x}dx=t\,dt$$

$$dx=\frac{t\,dt}{-e^{-2x}}$$

$$dx=-\frac{t\,dt}{t^2-1}\ \Big]$$

neodređeni integral

$$\int\sqrt{1+e^{-2x}}\,dx \overset{*}{=} \int -\frac{t^2}{t^2-1}\,dt=-\int\frac{t^2-1+1}{t^2-1}\,dt=-\int 1\,dt-\int\frac{1}{t^2-1}\,dt=$$

$$=-t-\frac{1}{2}\ln\left|\frac{t-1}{t+1}\right|=-\sqrt{1+e^{-2x}}-\frac{1}{2}\ln\left|\frac{\sqrt{1+e^{-2x}}-1}{\sqrt{1+e^{-2x}}+1}\right|$$

$$I=\left(-\sqrt{1+e^{-2x}}-\frac{1}{2}\ln\left|\frac{\sqrt{1+e^{-2x}}-1}{\sqrt{1+e^{-2x}}+1}\right|\right)\Bigg|_0^2=-\sqrt{1+e^{-4}}-\frac{1}{2}\ln\frac{\sqrt{1+e^{-4}}-1}{\sqrt{1+e^{-4}}+1}+\sqrt{2}+\frac{1}{2}\ln\left|\frac{\sqrt{2}-1}{\sqrt{2}+1}\right|$$

8. (jul 2024.)

Rešiti diferencijalnu jednačinu $y^2dx+x(\sqrt{y^2-x^2}-y)dy=0$

$$y^2dx+x(\sqrt{y^2-x^2}-y)dy=0 \;/: y^2$$

$$dx+\frac{x}{y}\cdot\left(\sqrt{1-\frac{x^2}{y^2}}-1\right)dy=0 \;/: dy$$

$$\frac{dx}{dy}=\frac{x}{y}\left(1-\sqrt{1-\frac{x^2}{y^2}}\right)$$

$$x'=\frac{x}{y}\left(1-\sqrt{1-\frac{x^2}{y^2}}\right)$$

$$z=\frac{x}{y} \qquad z=z(y)$$

$$x'=z'y+z$$

$$z'y+z=z\cdot\left(1-\sqrt{1-z^2}\right)$$

$$z'y+\cancel{z}=\cancel{z}-z\sqrt{1-z^2}$$

$$\frac{dz}{z\sqrt{1-z^2}}=-\frac{dy}{y} \;/\int$$

$$\int\frac{dz}{z\sqrt{1-z^2}}=\int\frac{z\,dz}{z^2\sqrt{1-z^2}}=$$

$$1-z^2=t^2$$

$$-2z\,dz=2t\,dt$$

$$-z\,dz=t\,dt$$

$$=\int\frac{-\cancel{t}\,dt}{(1-t^2)\cdot\cancel{t}}=\int\frac{dt}{t^2-1}=\frac{1}{2}\ln\left|\frac{t-1}{t+1}\right|$$

$$=\frac{1}{2}\ln\left|\frac{\sqrt{1-z^2}-1}{\sqrt{1-z^2}+1}\right|$$

$$\int\frac{dz}{z\sqrt{1-z^2}}=-\int\frac{dy}{y}$$

$$\frac{1}{2}\ln\left|\frac{\sqrt{1-z^2}-1}{\sqrt{1-z^2}+1}\right|=c-\ln|y|$$

$$\frac{1}{2}\ln\left|\frac{\sqrt{1-\left(\frac{x}{y}\right)^2}-1}{\sqrt{1-\left(\frac{x}{y}\right)^2}+1}\right|+\ln|y|=c$$

$$\ln\left|y\cdot\left(\frac{\sqrt{1-\frac{x^2}{y^2}}-1}{\sqrt{1-\frac{x^2}{y^2}}+1}\right)^{1/2}\right|=c$$

$$y\cdot\sqrt{\frac{\sqrt{1-\frac{x^2}{y^2}}-1}{\sqrt{1-\frac{x^2}{y^2}}+1}}=e^{c}=c_1$$

$$\boxed{y\cdot\sqrt{\frac{\sqrt{y^2-x^2}-1}{\sqrt{y^2-x^2}+1}}=c_1}$$