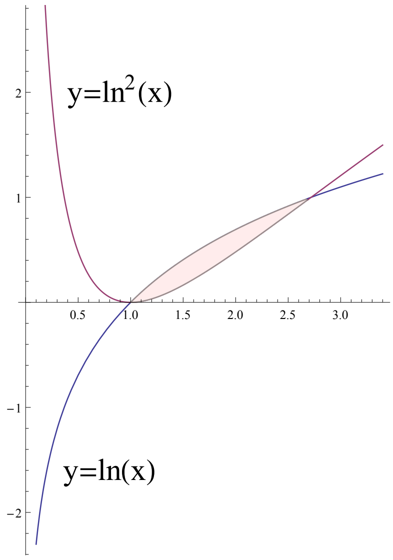
Univerzitet u Beogradu 28.04.2015. године

Mašinski fakultet

Katedra za Matematiku

Rešenje petog domaćeg zadatka iz Matematike 2

1. Površina koju treba izračunati prikazana je na slici 1. Prvo je potrebno naći presek krivih $y = \ln x$ и $y = \ln^2 x$. Iz $\ln x = \ln^2 x$ imamo $\ln^2 x - \ln x = 0$, odnosno $\ln x(\ln x - 1) = 0$, pa je $\ln x = 0$ ili $\ln x - 1 = 0$, odakle sledi $x = 1$ ili $x = e$. Kada se odrede odgovarajuće vrednosti y, dobijamo da su presečne tačke $(1, 0)$ и $(e, 1)$.



Slika 1: Površina koju treba izračunati u zadatku 1.

Ovde je $y = \ln x$ „горња", a $y = \ln^2 x$ „доња" крива. Tražena površina se dobija:

$$P = \int_1^e (\ln x - \ln^2 x)\,\mathrm{d}x = \int_1^e \ln x\,\mathrm{d}x - \int_1^e \ln^2 x\,\mathrm{d}x = I_1 - I_2$$

Izračunajmo I_1, parcijalnom gde smo odabrali podintegralne funkcije na sledeći način:

$$u = \ln x \Rightarrow \mathrm{d}u = \frac{1}{x}\mathrm{d}x, \quad v = \int dx \Rightarrow v = x$$

$$I_1 = \int_1^e \ln x\,\mathrm{d}x = x\ln x\Big|_1^e - \int_1^e \mathrm{d}x = e - (e-1) = 1$$

slično i za I_2, odaberimo podintegralne funkcije:

$$u = \ln^2 x \Rightarrow \mathrm{d}u = \frac{2\ln x}{x}\mathrm{d}x, \quad v = \int dx \Rightarrow v = x$$

$$I_2 = \int_1^e \ln^2 x\,\mathrm{d}x = x\ln^2 x\Big|_1^e - 2\int_1^e \ln x\,\mathrm{d}x = e - 2I_1 = e - 2$$

Konačno se dobija:

$$P = I_1 - I_2 = 1 - (e-2) = 3 - e$$

2. Neka je

$$I = \int_{-1}^{1} \frac{\mathrm{d}x}{(x+2)^2(x^2+1)}.$$

Razlomak u podintegralnom izrazu predstavlja pravu racionalnu funkciju i važi:

$$\frac{1}{(x+2)^2(x^2+1)} \equiv \frac{A}{x+2} + \frac{B}{(x+2)^2} + \frac{Cx+D}{x^2+1}.$$

Odnosno dve racionalne funkcije su identički jednake ako su im identički jednaki imenioci i brojioci:

$$\begin{aligned}
1 &\equiv A(x+2)(x^2+1) + B(x^2+1) + (Cx+D)(x+2)^2 \\
&\equiv A(x^3+2x^2+x+2) + B(x^2+1) + (Cx+D)(x^2+4x+4) \\
&\equiv Ax^3 + 2Ax^2 + Ax + 2A + Bx^2 + B + Cx^3 + 4Cx^2 + 4Cx + Dx^2 + 4Dx + 4D \\
&\equiv (A+C)x^3 + (2A+B+4C+D)x^2 + (A+4C+4D)x + (2A+B+4D)
\end{aligned}$$

odakle se dobija sledeći sistem jednačina:

$$\begin{array}{rcccccccl}
A & + & & & C & & & = & 0 \\
2A & + & \boxed{B} & + & 4C & + & D & = & 0 \quad {\scriptstyle -1} \\
A & + & & & 4C & + & 4D & = & 0 \\
2A & + & B & + & & & 4D & = & 1 \quad {\scriptstyle +}
\end{array}$$

$$\sim$$

$$\begin{array}{rcccccl}
\boxed{A} & + & & C & & = & 0 \quad {\scriptstyle -1} \\
A & + & & 4C & + \; 4D & = & 0 \quad {\scriptstyle +} \\
 & & - & 4C & + \; 3D & = & 1
\end{array}$$

$$\sim$$

$$\begin{array}{rcccl}
\boxed{3C} & + & 4D & = & 0 \quad {\scriptstyle \frac{4}{3}} \\
-4C & + & 3D & = & 1 \quad {\scriptstyle +}
\end{array}$$

$$\sim$$

$$D = \frac{3}{25}$$

Vraćajući vrednost za D u izbačene jednačine dobija se $C = -\frac{4}{25}$, $A = \frac{4}{25}$ i $B = \frac{1}{5}$. Osnovni integrali racionalne funkcije koji se dobijaju su:

$$\begin{aligned}
I &= \int_{-1}^{1} \left(\frac{\frac{4}{25}}{x+2} + \frac{\frac{1}{5}}{(x+2)^2} + \frac{-\frac{4}{25}x + \frac{3}{25}}{x^2+1} \right) \mathrm{d}x \\
&= \frac{4}{25} \int_{-1}^{1} \frac{\mathrm{d}x}{x+2} + \frac{1}{5} \int_{-1}^{1} \frac{\mathrm{d}x}{(x+2)^2} - \frac{2}{25} \int_{-1}^{1} \frac{2x\mathrm{d}x}{x^2+1} + \frac{3}{25} \int_{-1}^{1} \frac{\mathrm{d}x}{x^2+1} \\
&= \frac{4}{25} I_1 + \frac{1}{5} I_2 - \frac{2}{25} I_3 + \frac{3}{25} I_4
\end{aligned}$$

Izračunajmo ih redom:

$$I_1 = \int\limits_{-1}^{1} \frac{\mathrm{d}x}{x+2} = \ln|x+2|\Big|_{-1}^{1} = \ln 3 - ln1 = \ln 3$$

Posle smene $x+2=t \left\langle \begin{matrix} x=1 \Rightarrow t=3 \\ \\ x=-1 \Rightarrow t=1 \end{matrix} \right. \Rightarrow \mathrm{d}x = \mathrm{d}t$:

$$I_2 = \int\limits_{-1}^{1} \frac{\mathrm{d}x}{(x+2)^2} = \int\limits_{1}^{3} \frac{\mathrm{d}t}{t^2} = -\frac{1}{t}\Big|_{1}^{3} = -\frac{1}{3} + 1 = \frac{2}{3}$$

Kako je podintegralna funkcija u I_3 neparna, a interval integracije simetričan, sledi:

$$I_3 = \int\limits_{-1}^{1} \frac{2x\mathrm{d}x}{x^2+1} = 0$$

$$I_4 = \int\limits_{-1}^{1} \frac{\mathrm{d}x}{x^2+1} = \operatorname{arctg} x\Big|_{-1}^{1} = \operatorname{arctg} 1 - \operatorname{arctg}(-1) = \frac{\pi}{4} - \left(-\frac{\pi}{4}\right) = \frac{\pi}{2}$$

Konačno:

$$I = \frac{4}{25} \cdot \ln 3 + \frac{1}{5} \cdot \frac{2}{3} - \frac{2}{25} \cdot 0 + \frac{3}{25} \cdot \frac{\pi}{2} = \frac{24\ln 3 + 9\pi + 20}{150}$$

3. Ako se odaberu podintegralne funkcije na sledeći način:

$$u = \operatorname{arctg} x \Rightarrow \mathrm{d}u = \frac{\mathrm{d}x}{1+x^2}, \quad v = \int \mathrm{d}x \Rightarrow v = x$$

tada se može iskoristiti parcijalna integracija:

$$\int\limits_{0}^{1} \operatorname{arctg} x \,\mathrm{d}x = x \operatorname{arctg} x\Big|_{0}^{1} - \int\limits_{0}^{1} \frac{x\mathrm{d}x}{1+x^2} = \operatorname{arctg} 1 - 0 - \frac{1}{2}\int\limits_{0}^{1} \frac{2x\mathrm{d}x}{1+x^2} = \frac{\pi}{4} - \frac{1}{2}I_1$$

Posle smene $1+x^2=t \left\langle \begin{matrix} x=1 \Rightarrow t=2 \\ \\ x=0 \Rightarrow t=1 \end{matrix} \right. \Rightarrow 2x\,\mathrm{d}x = \mathrm{d}t$ jednostavno se dobija rešenje I_1:

$$I_1 = \int\limits_{0}^{1} \frac{2x\mathrm{d}x}{1+x^2} = \int\limits_{1}^{2} \frac{\mathrm{d}t}{t} = \ln|t|\Big|_{1}^{2} = \ln 2$$

Konačno tražena vrednost jednaka je:

$$I = \frac{\pi}{4} - \frac{\ln 2}{2}$$

4. Pre parcijalne integracije transformišimo podintegralnu funkciju na sledeći način:

$$I_n = \int\limits_0^{\frac{\pi}{2}} \cos^n x \, \mathrm{d}x = \int\limits_0^{\frac{\pi}{2}} \cos^{n-1} x \cos x \, \mathrm{d}x$$

i ako se odaberu na sledeći način podintegralne funkcije:

$$u = \cos^{n-1} x \Rightarrow \mathrm{d}u = (n-1)\cos^{n-2} x\,(-\sin x)\,, \quad v = \int \cos x \, \mathrm{d}x \Rightarrow v = \sin x$$

$$\begin{aligned}
I_n &= \cos^{n-1} x \sin x \Big|_0^{\frac{\pi}{2}} + (n-1)\int\limits_0^{\frac{\pi}{2}} \cos^{n-2} x \sin^2 x \, \mathrm{d}x \\
&= 0 + (n-1)\int\limits_0^{\frac{\pi}{2}} \cos^{n-2} x(1-\cos^2 x) \, \mathrm{d}x \\
&= (n-1)\left(\int\limits_0^{\frac{\pi}{2}} \cos^{n-2} x \, \mathrm{d}x - \int\limits_0^{\frac{\pi}{2}} \cos^n x \, \mathrm{d}x\right) \\
&= (n-1)(I_{n-2} - I_n)
\end{aligned}$$

odakle sledi:

$$I_n = \frac{n-1}{n} I_{n-2}$$

Primenjujući rekurentnu formulu:

$$\begin{aligned}
I_n &= \frac{n-1}{n} I_{n-2} = \frac{n-1}{n}\cdot\frac{n-3}{n-2} I_{n-4} = \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4} I_{n-6} = \cdots \\
&= \begin{cases} \dfrac{n-1}{n}\cdot\dfrac{n-3}{n-2}\cdots\dfrac{2}{3} I_1 & , n \in 2\mathbb{N}+1 \\ \dfrac{n-1}{n}\cdot\dfrac{n-3}{n-2}\cdots\dfrac{3}{4} I_2 & , n \in 2\mathbb{N} \end{cases}
\end{aligned}$$

gde su terminalni integrali:

$$I_1 = \int\limits_0^{\frac{\pi}{2}} \cos x \, \mathrm{d}x = \sin x \Big|_0^{\frac{\pi}{2}} = \sin\frac{\pi}{2} - \sin 0 = 1 - 0 = 1$$

ili

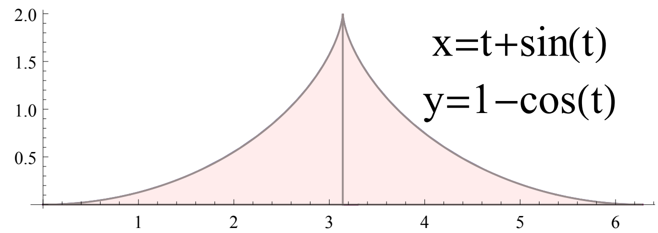
$$\begin{aligned}
I_2 &= \int\limits_0^{\frac{\pi}{2}} \cos^2 x \, \mathrm{d}x = \int\limits_0^{\frac{\pi}{2}} \frac{1+\cos 2x}{2} \, \mathrm{d}x = \frac{1}{2}\left(x + \frac{1}{2}\sin 2x\right)\Big|_0^{\frac{\pi}{2}} \\
&= \frac{1}{2}\left(\frac{\pi}{2} - 0 + \sin\pi - \sin 0\right) = \frac{1}{2}\left(\frac{\pi}{2} - 0 + 0 - 0\right) = \frac{1}{2}\cdot\frac{\pi}{2} = \frac{\pi}{4}
\end{aligned}$$

Konačno se može napisati:

$$
\begin{aligned}
I_n &= \begin{cases} \dfrac{n-1}{n}\cdot\dfrac{n-3}{n-2}\cdots\dfrac{2}{3}\cdot 1 & , n\in 2\mathbb{N}+1 \\ \dfrac{n-1}{n}\cdot\dfrac{n-3}{n-2}\cdots\dfrac{3}{4}\cdot\dfrac{\pi}{4} & , n\in 2\mathbb{N} \end{cases} \\
&= \begin{cases} \dfrac{(n-1)!!}{n!!} & , n\in 2\mathbb{N}+1 \\ \dfrac{(n-1)!!}{n!!}\cdot\dfrac{\pi}{2} & , n\in 2\mathbb{N} \end{cases}
\end{aligned}
$$

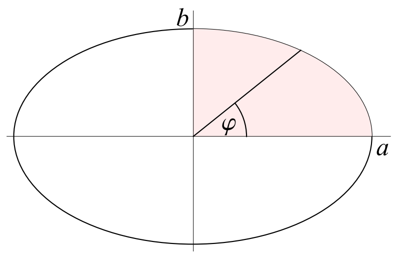
5. Površina koju treba izračunati prikazana je na slici 2. Važi:

$$
\begin{aligned}
P &= \int_0^{2\pi} y\,\mathrm{d}x = \int_0^{2\pi}(1-\cos t)(1+\cos t)\,\mathrm{d}t = \int_0^{2\pi}(1-\cos^2 t)\,\mathrm{d}t = \int_0^{2\pi}\left(1-\frac{1+\cos 2t}{2}\right)\,\mathrm{d}t \\
&= \frac{1}{2}\int_0^{2\pi}(1-\cos 2t)\,\mathrm{d}t = \frac{1}{2}\left(t-\frac{1}{2}\sin 2t\right)\Bigg|_0^{2\pi} = \frac{1}{2}\cdot 2\pi = \pi
\end{aligned}
$$



Slika 2: Površina omeđena krivima $y = 1-\cos t$, $x = t+\sin t$ na intervalu $(0, 2\pi)$

6. Prelazeći na uopštene polarne koordinate $x = ar\cdot\cos\varphi$, $y = br\cdot\sin\varphi$ vidi sliku 3: jednačina



Slika 3: Elipsa sa polarnim koordinatama.

elipse postaje:

$$\frac{a^2r^2\cos^2\varphi}{a^2}+\frac{b^2r^2\sin^2\varphi}{b^2}=1$$

odakle je

$$r^2 = 1$$

Zbog očite simetrije, traženu površinu možemo izračunati kao četiri površine osenčenog dela na slici 3 prema formuli za površinu ravnog lika u polarnim koordinatama:

$$P = 4 \cdot \frac{1}{2} \int_0^{\pi/2} ab \cdot r^2 \, \mathrm{d}\varphi = 2ab \int_0^{\pi/2} \mathrm{d}\varphi = 2ab \cdot \varphi \Big|_0^{\pi/2} = 2ab \left(\frac{\pi}{2} - 0 \right) = ab\pi$$

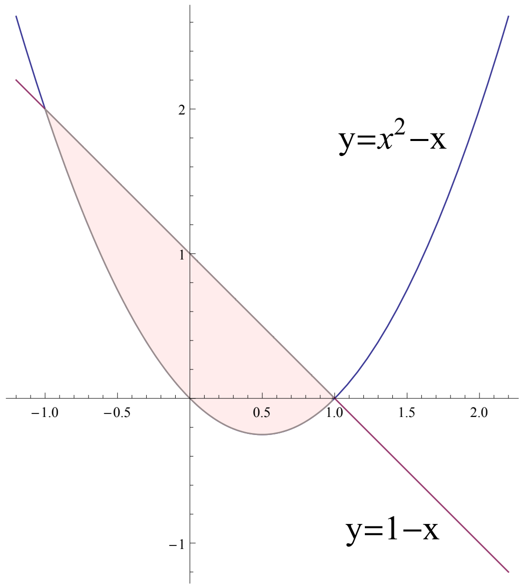
7. Jednačina normale na krivu $y = y(x)$ u tački $M(x_0, y_0)$ je

$$y - y_0 = -\frac{1}{y'(x_0)}(x - x_0)$$

Kako je $y = x(x-1) = x^2 - x$, to je $y' = 2x - 1$, pa je u datoj tački $x_0 = 1$ i $y_0 = 0$ jednačina normale:

$$y - 0 = -\frac{1}{2 \cdot 1 - 1}(x - 1) \Rightarrow y = 1 - x$$

kako je to prikazano na slici:



Presečne tačke parabole i normale nalazimo rešavanjem sistema jednačina:

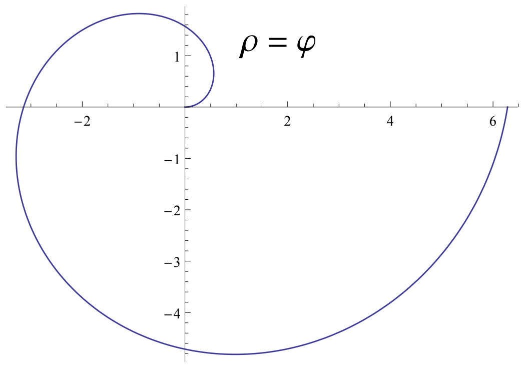
$$y = x^2 - x, \quad y = 1 - x$$

pa su x-koordinate presečnih tačaka -1 i 1. Tražena površina je:

$$\begin{aligned} P &= \int_{-1}^{1} \left(1 - x - \left(x^2 - x\right)\right) \, \mathrm{d}x = 2 \int_0^1 \left(1 - x^2\right) \, \mathrm{d}x = 2 \left(x - \frac{x^3}{3} \right) \Bigg|_0^1 \\ &= 2\left(\left(1 - 1^3/3\right) - \left(0 - 0^3/3\right)\right) = 2 \cdot \frac{2}{3} = \frac{4}{3} \end{aligned}$$

Druga jednakost u gornjem izvođenju važi jer je podintegralna funkcija parna, a interval integracije simetričan u odnosu na koordinatni početak.

8. Polarna jednačina $\rho = \varphi$, $\varphi \in [0, 2\pi]$ je jednačina spirale na slici:



Njena dužina luka je:

$$L = \int_0^{2\pi} \sqrt{\rho^2 + \rho'^2}\,\mathrm{d}\varphi = \int_0^{2\pi} \sqrt{\varphi^2 + 1}\,\mathrm{d}\varphi$$

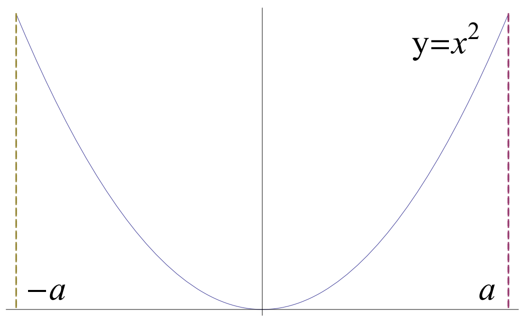
Na osnovu rađenog neodređenog integrala:

$$\int \sqrt{x^2 + a^2} dx = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2}\ln(x + \sqrt{x^2 + a^2}) + c$$

je

$$\begin{aligned} L &= \frac{2\pi}{2}\sqrt{(2\pi)^2 + 1} + \frac{1}{2}\ln\left(2\pi + \sqrt{(2\pi)^2 + 1}\right) - \left(\frac{0}{2}\sqrt{0^2 + 1} + \frac{1}{2}\ln\left(0 + \sqrt{0^2 + 1}\right)\right) \\ &= \pi\sqrt{4\pi^2 + 1} + \frac{1}{2}\ln\left(2\pi + \sqrt{4\pi^2 + 1}\right) \end{aligned}$$

9. Dužina luka parabole $y = x^2$ za $x \in [-a, a]$, vidi sliku, je:

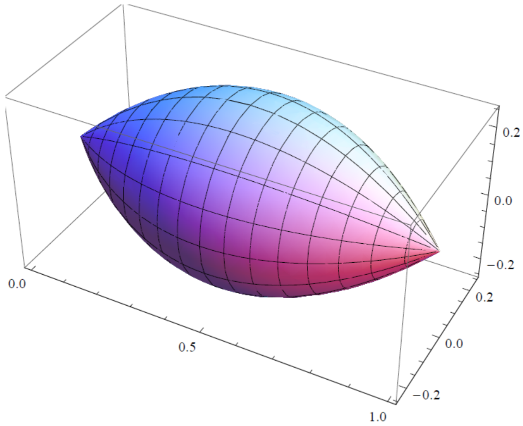


$$L = \int_{-a}^{a} \sqrt{1 + (y')^2}\,\mathrm{d}x = \int_{-a}^{a} \sqrt{1 + (2x)^2}\,\mathrm{d}x = 2\int_0^{a} \sqrt{1 + (2x)^2}\,\mathrm{d}x$$

Uvodeći smenu $2x = t \left\langle \begin{array}{l} x=a \Rightarrow t=2a \\ \\ x=0 \Rightarrow t=0 \end{array} \right. \Rightarrow 2\mathrm{d}x = \mathrm{d}t$ dobijamo $L = \int_0^{2a} \sqrt{1 + t^2}\,\mathrm{d}t$. Na osnovu korišćenog „tabličnog” integrala iz prethodnog zadatka:

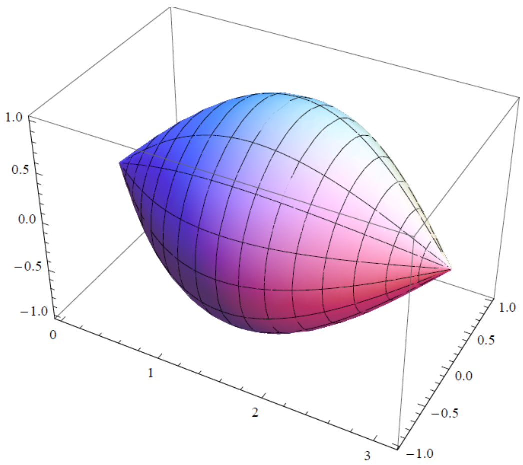
$$L = \frac{2a}{2}\sqrt{(2a)^2 + 1} + \frac{1}{2}\ln\left(2a + \sqrt{(2a)^2 + 1}\right) = a\sqrt{4a^2 + 1} + \frac{1}{2}\ln\left(2a + \sqrt{4a^2 + 1}\right)$$

10. Rotacijom parabole $y = x - x^2$ oko x-ose za $x \in [0, 1]$ dobija se telo (vidi sliku) čija je zapremina:



$$V = \pi \int\limits_0^1 y^2 \, \mathrm{d}x = \pi \int\limits_0^1 \left(x^2 - 2x^3 + x^4\right) \, \mathrm{d}x = \pi \left(\frac{x^3}{3} - 2\frac{x^4}{4} + \frac{x^5}{5} \right) \Bigg|_0^1 = \pi \left(\frac{1}{3} - \frac{1}{2} + \frac{1}{5} \right) = \frac{\pi}{30}$$

11. Telo koje se dobija rotacijom oko x-ose krive $y = \sin x$, $x \in [0, \pi]$ je prikazano na slici 4.



Slika 4: Telo koje nastaje rotacijom krive $y = \sin x$, $x \in [0, \pi]$ oko x-ose.

Zapreminu dobijamo jednostavno

$$V = \pi \int\limits_0^\pi y^2 \, \mathrm{d}x = \pi \int_0^\pi \sin^2 x \, \mathrm{d}x = \pi \int\limits_0^\pi \frac{1 - \cos 2x}{2} \, \mathrm{d}x = \frac{\pi}{2}$$

12. Integral je nesvojstven jer je podintegralna funkcija, označimo je sa $f(x)$, neograničena u okolini tačaka 1 i 2 koje nisu u oblasti definisanosti podintegralne funkcije. Sledi:

$$\begin{aligned}
I &= \int_0^1 f(x)\,\mathrm{d}x + \int_1^2 f(x)\,\mathrm{d}x + \int_2^3 f(x)\,\mathrm{d}x \\
&= \lim_{\varepsilon_1\to 0}\int_0^{1-\varepsilon_1} f(x)\,\mathrm{d}x + \lim_{\substack{\varepsilon_2\to 0\\ \varepsilon_3\to 0}}\int_{1+\varepsilon_2}^{2-\varepsilon_3} f(x)\,\mathrm{d}x + \lim_{\varepsilon_4\to 0}\int_{2+\varepsilon_4}^{3} f(x)\,\mathrm{d}x \\
&= \lim_{\varepsilon_1\to 0}\int_0^{1-\varepsilon_1} \frac{\mathrm{d}x}{\sqrt{(x-1)(x-2)}} + \lim_{\substack{\varepsilon_2\to 0\\ \varepsilon_3\to 0}}\int_{1+\varepsilon_2}^{2-\varepsilon_3} \frac{\mathrm{d}x}{\sqrt{-(x-1)(x-2)}} + \\
&\quad \lim_{\varepsilon_4\to 0}\int_{2+\varepsilon_4}^{3} \frac{\mathrm{d}x}{\sqrt{(x-1)(x-2)}}
\end{aligned}$$

Ako se iskoriste smene Ojlera, ili na neki drugi način, može se dobiti:

$$\begin{aligned}
\int \frac{dx}{\sqrt{(x-1)(x-2)}} &= \ln\left|3-2x-2\sqrt{(x-1)(x-2)}\right| + c \\
\int \frac{dx}{\sqrt{-(x-1)(x-2)}} &= -\arcsin(3-2x) + c
\end{aligned}$$

Tada se dobija:

$$\begin{aligned}
\lim_{\varepsilon_1\to 0}\int_0^{1-\varepsilon_1} \frac{\mathrm{d}x}{\sqrt{(x-1)(x-2)}} &= \lim_{\varepsilon_1\to 0} \ln\left|3-2x-2\sqrt{(x-1)(x-2)}\right|\Bigg|_0^{1-\varepsilon_1} \\
&= \lim_{\varepsilon_1\to 0}\Bigg[\ln\left|3-2(1-\varepsilon_1)-2\sqrt{(1-\varepsilon_1-1)(1-\varepsilon_1-2)}\right| - \\
&\qquad \ln\left|3-2\sqrt{2}\right|\Bigg] \\
&= \lim_{\varepsilon_1\to 0}\left[\ln\left|1-2\sqrt{\varepsilon_1(1+\varepsilon_1)}\right| - \ln\left|3-2\sqrt{2}\right|\right] = -\ln(3-2\sqrt{2})
\end{aligned}$$

$$\begin{aligned}
\lim_{\substack{\varepsilon_2\to 0\\ \varepsilon_3\to 0}}\int_{1+\varepsilon_2}^{2-\varepsilon_3} \frac{\mathrm{d}x}{\sqrt{-(x-1)(x-2)}} &= \lim_{\substack{\varepsilon_2\to 0\\ \varepsilon_3\to 0}}\Big[-\arcsin(3-2x)\Big]\Bigg|_{1+\varepsilon_2}^{2-\varepsilon_3} \\
&= \lim_{\substack{\varepsilon_2\to 0\\ \varepsilon_3\to 0}}\Big[-\arcsin(3-2(2-\varepsilon_3)) + \arcsin(3-2(1+\varepsilon_2))\Big] \\
&= -\arcsin(-1) + \arcsin(1) = \pi
\end{aligned}$$

$$\begin{aligned}
\lim_{\varepsilon_4\to 0}\int\limits_{2+\varepsilon_4}^{3}\frac{\mathrm{d}x}{\sqrt{(x-1)(x-2)}} &= \lim_{\varepsilon_4\to 0}\ln\left|3-2x-2\sqrt{(x-1)(x-2)}\right|\Bigg|_{2+\varepsilon_4}^{3}\\
&= \lim_{\varepsilon_4\to 0}\Bigg[\ln\left|3-2\cdot 3-2\sqrt{(3-1)(3-2)}\right|-\\
&\quad \ln\left|3-2(2+\varepsilon_4)-2\sqrt{(2+\varepsilon_4-1)(2+\varepsilon_4-2)}\right|\Bigg]\\
&= \lim_{\varepsilon_4\to 0}\left[\ln\left|-3-2\sqrt{2}\right|-\ln\left|-1-2\varepsilon_4-2\sqrt{\varepsilon_4(1+\varepsilon_4)}\right|\right]\\
&= \ln\left(3+2\sqrt{2}\right)
\end{aligned}$$

Konačno:

$$\begin{aligned}
I &= -\ln(3-2\sqrt{2})+\pi+\ln(3+2\sqrt{2})\\
&= \pi+\ln\frac{3+2\sqrt{2}}{3-2\sqrt{2}}\\
&= \pi+2\ln(3+2\sqrt{2})\\
&= \pi+4\ln(1+\sqrt{2})
\end{aligned}$$

13. Neka je $I_3=\lim\limits_{A\to+\infty}\int\limits_0^A x^3e^{-x}\mathrm{d}x$ i rešimo ga parcijalnim integracijama, gde je izabrano:

$$u=x^3\Rightarrow \mathrm{d}u=3x^2\mathrm{d}x,\quad v=\int e^{-x}\mathrm{d}x=-e^{-x}$$

tada je:

$$\begin{aligned}
I_3 &= \lim_{A\to+\infty}\left[x^3e^{-x}\Big|_{x=A}^{x=0}+3\int\limits_0^A x^2e^{-x}\mathrm{d}x\right]\\
&= \lim_{A\to+\infty}\left[0-A^3e^{-A}+3\int\limits_0^A x^2e^{-x}\mathrm{d}x\right]\\
&= -\lim_{A\to+\infty}\frac{A^3}{e^A}+3\lim_{A\to+\infty}\int\limits_0^A x^2e^{-x}\mathrm{d}x
\end{aligned}$$

Kako je e^A brže rastuća funkcija od bilo kog polinoma, tada je $\lim\limits_{A\to+\infty}\dfrac{A^3}{e^A}=0$. Konačno se dobija:

$$I_3=0+3\lim_{A\to+\infty}\int\limits_0^A x^2e^{-x}\mathrm{d}x=3\lim_{A\to+\infty}\int\limits_0^A x^2e^{-x}\mathrm{d}x=3\cdot I_2$$

Primenjujući rekurentnu vezu vrednost integrala je:

$$I_3=3\cdot I_2=3\cdot 2\cdot I_1=3\cdot 2\cdot 1\cdot I_0$$

gde je:

$$I_0 = \lim_{A\to+\infty} \int\limits_0^A x^0 e^{-x} \mathrm{d}x = \lim_{A\to+\infty} \int\limits_0^A e^{-x} \mathrm{d}x = \lim_{A\to+\infty} e^{-x} \Big|_{x=A}^{x=0} = 1 - \lim_{A\to+\infty} \frac{1}{e^A} = 1$$

Sledi da je:

$$I_3 = \int\limits_0^{+\infty} x^3 e^{-x} \mathrm{d}x = 6$$

14. Ako dati integral posmatramo kao nesvojstveni integral:

$$I = \int\limits_{-\infty}^{+\infty} \sin x s dx = \lim_{\substack{A_1\to+\infty \\ A_2\to+\infty}} \int\limits_{-A_2}^{A_1} \sin x \ \mathrm{d}x = \lim_{\substack{A_1\to+\infty \\ A_2\to+\infty}} (-\cos A_1 + \cos A_2)$$

dobijamo da I ne konvergira jer granična vrednost poslednjeg izraza ne postoji. Na primer, ako izaberemo $A_1 = A_2$, dobijamo:

$$I = \lim_{A_1\to+\infty} (-\cos A_1 + \cos A_1) = 0$$

dok izborom $A_2 = \pi + A_1$, $A_1 = 2k\pi$, gde je $k \in \mathbb{N}$, dobijamo:

$$I = \lim_{A_1\to+\infty} (-\cos A_1 + \cos(\pi + A_1)) = -2 \lim_{k\to+\infty} \cos(2k\pi) = -2$$

Nekim drugim izborom može se dobiti neka treća vrednost. Međutim, postoji glavna vrednost integrala:

$$I = v.p. \int\limits_{-\infty}^{+\infty} \sin x \ \mathrm{d}x = \lim_{A\to+\infty} \int\limits_{-A}^{A} \sin x \ \mathrm{d}x = \lim_{A\to+\infty} (-\cos x) \Big|_{-A}^{A} = \lim_{A\to+\infty} (-\cos A + \cos A) = 0$$

15. Kako je $D_f = \mathbb{R} \times (-\infty, 0) \cup (0, +\infty)$ jednostavno nađimo potrebne izvode:

$$\frac{\partial z}{\partial x} = \frac{1}{1 + \left(\frac{x}{y}\right)^2} \cdot \frac{1}{y} = \frac{y}{y^2 + x^2}, \quad \frac{\partial z}{\partial y} = \frac{1}{1 + \left(\frac{x}{y}\right)^2} \cdot \frac{-x}{y^2} = \frac{-x}{y^2 + x^2}$$

Drugi parcijalni izvodi su:

$$\begin{aligned}
\frac{\partial^2 z}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = y \cdot \frac{-1}{(y^2 + x^2)^2} \cdot 2x = \frac{-2xy}{(y^2 + x^2)^2} \\
\frac{\partial^2 z}{\partial y^2} &= \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = -x \cdot \frac{-1}{(y^2 + x^2)^2} \cdot 2y = \frac{2xy}{(y^2 + x^2)^2} \\
\frac{\partial^2 z}{\partial x \partial y} &= \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{-1 \cdot (y^2 + x^2) + x \cdot 2x}{(y^2 + x^2)^2} = \frac{x^2 - y^2}{(y^2 + x^2)^2} \\
\frac{\partial^2 z}{\partial y \partial x} &= \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{1 \cdot (y^2 + x^2) - y \cdot 2y}{(y^2 + x^2)^2} = \frac{x^2 - y^2}{(y^2 + x^2)^2}
\end{aligned}$$

16. Da bi formirali matricu potrebni su sledeći prvi parcijalni izvodi:

$$\begin{aligned}
\frac{\partial u}{\partial x} &= 2\cdot(x+y+z)\cdot 1 + 2\cdot(x-y+z)\cdot 1 + 2\cdot(x+y-z)\cdot 1 = 2\,(3x+y+z)\\
\frac{\partial u}{\partial y} &= 2\cdot(x+y+z)\cdot 1 + 2\cdot(x-y+z)\cdot(-1) + 2\cdot(x+y-z)\cdot 1 = 2\,(x+3y-z)\\
\frac{\partial u}{\partial z} &= 2\cdot(x+y+z)\cdot 1 + 2\cdot(x-y+z)\cdot 1 + 2\cdot(x+y-z)\cdot(-1) = 2\,(x-y+3z)
\end{aligned}$$

i sledeći drugi parcijalni izvodi:

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2} &= 6\\
\frac{\partial^2 u}{\partial x\partial y} &= 2\\
\frac{\partial^2 u}{\partial x\partial z} &= 2\\
\frac{\partial^2 u}{\partial y^2} &= 6\\
\frac{\partial^2 u}{\partial y\partial z} &= -2\\
\frac{\partial^2 u}{\partial y^2} &= 6
\end{aligned}$$

Tražena matrica je:

$$H = \begin{bmatrix} 6 & 2 & 2\\ 2 & 6 & -2\\ 2 & -2 & 6 \end{bmatrix} = 2\begin{bmatrix} 3 & 1 & 1\\ 1 & 3 & -1\\ 1 & -1 & 3 \end{bmatrix}$$

Na uobičajen način nađimo rang matrice H:

$$\begin{bmatrix} 3 & 1 & \boxed{1}\\ 1 & 3 & -1\\ 1 & -1 & 3 \end{bmatrix} \sim \begin{bmatrix} 3 & 1 & \boxed{1}\\ 4 & \boxed{4} & 0\\ -8 & -4 & 0 \end{bmatrix} \sim \begin{bmatrix} 3 & 1 & \boxed{1}\\ 4 & \boxed{4} & 0\\ \boxed{-4} & 0 & 0 \end{bmatrix}$$

Odakle je $\rho\,(H) = 3$.

17. Nađimo:

$$\frac{\partial r}{\partial x} = \frac{\partial}{\partial x}\left(\sqrt{x^2+y^2+z^2}\right) = \frac{1}{2\sqrt{x^2+y^2+z^2}}\cdot 2x = \frac{x}{r}$$

slično se nalaze i ostali parcijalni izvodi:

$$\begin{aligned}
\frac{\partial r}{\partial y} &= \frac{\partial}{\partial y}\left(\sqrt{x^2+y^2+z^2}\right) = \frac{1}{2\sqrt{x^2+y^2+z^2}}\cdot 2y = \frac{y}{r}\\
\frac{\partial r}{\partial z} &= \frac{\partial}{\partial z}\left(\sqrt{x^2+y^2+z^2}\right) = \frac{1}{2\sqrt{x^2+y^2+z^2}}\cdot 2z = \frac{z}{r}
\end{aligned}$$

Iskoristimo poznate formule:

$$\begin{aligned}
\frac{\partial F}{\partial x} &= \frac{\partial f}{\partial r}\cdot\frac{\partial r}{\partial x} = \frac{\partial}{\partial r}\left(r^{\alpha}\right)\cdot\frac{x}{r} = \alpha r^{\alpha-1}\cdot\frac{x}{r} = \alpha x r^{\alpha-2}\\
\frac{\partial F}{\partial y} &= \frac{\partial f}{\partial r}\cdot\frac{\partial r}{\partial y} = \frac{\partial}{\partial r}\left(r^{\alpha}\right)\cdot\frac{y}{r} = \alpha r^{\alpha-1}\cdot\frac{y}{r} = \alpha y r^{\alpha-2}\\
\frac{\partial F}{\partial z} &= \frac{\partial f}{\partial r}\cdot\frac{\partial r}{\partial z} = \frac{\partial}{\partial r}\left(r^{\alpha}\right)\cdot\frac{z}{r} = \alpha r^{\alpha-1}\cdot\frac{z}{r} = \alpha z r^{\alpha-2}
\end{aligned}$$

Analizirajući prethodne izvode, može se uočiti da se parcijalni izvodi po y i z mogu odmah napisati koristeći se parcijalnim izvodom po x. Funkcija je takvog oblika - simetrična po svim promenljivama. Iskoristimo ovu činjenicu za naredna računanja.

$$\begin{aligned}
\frac{\partial^2 F}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial x}\right)\\
&= \frac{\partial}{\partial x}\left(\alpha x r^{\alpha-1}\right)\\
&= \alpha\frac{\partial}{\partial x}\left(x r^{\alpha-1}\right)\\
&= \alpha\left[\frac{\partial}{\partial x}(x)\cdot r^{\alpha-2} + x\cdot\frac{\partial}{\partial x}\left(r^{\alpha-2}\right)\right]\\
&= \alpha\left[1\cdot r^{\alpha-2} + x\cdot\frac{\partial}{\partial r}\left(r^{\alpha-2}\right)\cdot\frac{\partial r}{\partial x}\right]\\
&= \alpha\left[r^{\alpha-2} + x\cdot(\alpha-2)\, r^{\alpha-3}\cdot\frac{x}{r}\right]\\
&= \alpha\left[r^{\alpha-2} + (\alpha-2)\cdot x^2\cdot r^{\alpha-4}\right]
\end{aligned}$$

Napišimo i ostale parcijalne izvode koristeći dobijeni parcijalni izvod:

$$\begin{aligned}
\frac{\partial^2 F}{\partial y^2} &= \alpha\left[r^{\alpha-2} + (\alpha-2)\cdot y^2\cdot r^{\alpha-4}\right]\\
\frac{\partial^2 F}{\partial z^2} &= \alpha\left[r^{\alpha-2} + (\alpha-2)\cdot z^2\cdot r^{\alpha-4}\right]
\end{aligned}$$

Sledi da je traženi zbir:

$$\begin{aligned}
\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} + \frac{\partial^2 F}{\partial z^2} &= \alpha\left[r^{\alpha-2} + (\alpha-2)\cdot x^2\cdot r^{\alpha-4}\right] +\\
&\quad\ \alpha\left[r^{\alpha-2} + (\alpha-2)\cdot y^2\cdot r^{\alpha-4}\right] +\\
&\quad\ \alpha\left[r^{\alpha-2} + (\alpha-2)\cdot z^2\cdot r^{\alpha-4}\right]\\
&= \alpha\left\{3\cdot r^{\alpha-2} + (\alpha-2)\cdot r^{\alpha-4}\cdot\left[x^2+y^2+z^2\right]\right\}\\
&= \alpha\left[3\cdot r^{\alpha-2} + (\alpha-2)\cdot r^{\alpha-4}\cdot r^2\right]\\
&= \alpha\left[3\cdot r^{\alpha-2} + (\alpha-2)\cdot r^{\alpha-2}\right]\\
&= \alpha\cdot r^{\alpha-2}\left(3+\alpha-2\right)\\
&= (\alpha+1)\cdot\alpha\cdot r^{\alpha-2}
\end{aligned}$$

18. Označimo sa $u = u(x,y) = ax + by$, $v = v(x,y) = \sqrt{x^2+y^2}$ tada je $F(u,v) = f(x,y)$. Izračunajmo:

$$\begin{aligned}
\frac{\partial u}{\partial x} &= a \\
\frac{\partial u}{\partial y} &= b \\
\frac{\partial v}{\partial x} &= \frac{1}{2\sqrt{x^2+y^2}} \cdot 2x = \frac{x}{\sqrt{x^2+y^2}} = \frac{x}{v} \\
\frac{\partial v}{\partial y} &= \frac{1}{2\sqrt{x^2+y^2}} \cdot 2y = \frac{y}{\sqrt{x^2+y^2}} = \frac{y}{v}
\end{aligned}$$

Ako iskoristimo prethodno dobijene parcijalne izvode dobijamo:

$$\begin{aligned}
\frac{\partial f}{\partial x} &= \frac{\partial F}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial F}{\partial v} \cdot \frac{\partial v}{\partial x} = \frac{\partial F}{\partial u} \cdot a + \frac{\partial F}{\partial v} \cdot \frac{x}{v} \\
\frac{\partial f}{\partial y} &= \frac{\partial F}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial F}{\partial v} \cdot \frac{\partial v}{\partial y} = \frac{\partial F}{\partial u} \cdot b + \frac{\partial F}{\partial v} \cdot \frac{y}{v}
\end{aligned}$$

Izračunajmo:

$$\begin{aligned}
\frac{\partial^2 f}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u} \cdot a + \frac{\partial F}{\partial v} \cdot \frac{x}{v}\right) \\
&= a \cdot \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u}\right) + \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial v}\right) \cdot \frac{x}{v} + \frac{\partial F}{\partial v} \cdot \frac{\partial}{\partial x}\left(\frac{x}{v}\right) \\
&= a \cdot \left[\frac{\partial}{\partial u}\left(\frac{\partial F}{\partial u}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial F}{\partial u}\right) \cdot \frac{\partial v}{\partial x}\right] \\
&\quad + \left[\frac{\partial}{\partial u}\left(\frac{\partial F}{\partial v}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial F}{\partial v}\right) \cdot \frac{\partial v}{\partial x}\right] \cdot \frac{x}{v} \\
&\quad + \frac{\partial F}{\partial v} \cdot \left[\frac{\partial}{\partial u}\left(\frac{x}{v}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{x}{v}\right) \cdot \frac{\partial v}{\partial x}\right] \\
&= a \cdot \left(\frac{\partial^2 F}{\partial u^2} \cdot a + \frac{\partial^2 F}{\partial v \partial u} \cdot \frac{x}{v}\right) + \left(\frac{\partial^2 F}{\partial u \partial v} \cdot a + \frac{\partial^2 F}{\partial v^2} \cdot \frac{x}{v}\right) \cdot \frac{x}{v} + \frac{\partial F}{\partial v} \cdot \left(0 + \frac{-x}{v^2} \cdot \frac{x}{v}\right)
\end{aligned}$$

Konačno se dobija:

$$\frac{\partial^2 f}{\partial x^2} = a^2 \cdot \frac{\partial^2 F}{\partial u^2} + 2a \cdot \frac{x}{v} \cdot \frac{\partial^2 F}{\partial u \partial v} + \frac{x^2}{v^2} \cdot \frac{\partial^2 F}{\partial v^2} - \frac{x^2}{v^3} \cdot \frac{\partial F}{\partial v}$$

Slično se može pokazati da je:

$$\frac{\partial^2 f}{\partial y^2} = b^2 \cdot \frac{\partial^2 F}{\partial u^2} + 2b \cdot \frac{y}{v} \cdot \frac{\partial^2 F}{\partial u \partial v} + \frac{y^2}{v^2} \cdot \frac{\partial^2 F}{\partial v^2} - \frac{y^2}{v^3} \cdot \frac{\partial F}{\partial v}$$

$$
\begin{aligned}
\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2} &= a^2\cdot\frac{\partial^2 F}{\partial u^2}+2a\cdot\frac{x}{v}\cdot\frac{\partial^2 F}{\partial u\partial v}+\frac{x^2}{v^2}\cdot\frac{\partial^2 F}{\partial v^2}-\frac{x^2}{v^3}\cdot\frac{\partial F}{\partial v}+\\
&\quad b^2\cdot\frac{\partial^2 F}{\partial u^2}+2b\cdot\frac{y}{v}\cdot\frac{\partial^2 F}{\partial u\partial v}+\frac{y^2}{v^2}\cdot\frac{\partial^2 F}{\partial v^2}-\frac{y^2}{v^3}\cdot\frac{\partial F}{\partial v}\\
&= \left(a^2+b^2\right)\cdot\frac{\partial^2 F}{\partial u^2}+\frac{2}{v}\cdot(ax+by)\cdot\frac{\partial^2 F}{\partial u\partial v}+\\
&\quad \frac{1}{v^2}\cdot\left(x^2+y^2\right)\cdot\frac{\partial^2 F}{\partial v^2}-\frac{1}{v^3}\cdot\left(x^2+y^2\right)\cdot\frac{\partial F}{\partial v}\\
&= \left(a^2+b^2\right)\cdot\frac{\partial^2 F}{\partial u^2}+2\frac{u}{v}\cdot\frac{\partial^2 F}{\partial u\partial v}+\frac{\partial^2 F}{\partial v^2}-\frac{1}{v}\cdot\frac{\partial F}{\partial v}
\end{aligned}
$$

19. Pogodno je napisati da je:

$$
\begin{aligned}
z &= \varphi(u)+\psi(v)=f(u,v)\\
u &= x+ay=u(x,y)\\
v &= x-ay=v(x,y)
\end{aligned}
$$

gde je $a\in\mathbf{R}$. Iz postavke zadatka jednostavno se dobijaju sledeći potrebni izvodi:

$$\frac{\partial u}{\partial x}=1,\quad \frac{\partial v}{\partial x}=1$$

slično:

$$\frac{\partial u}{\partial y}=a,\quad \frac{\partial v}{\partial y}=-a$$

a drugi parcijalni izvodi su jednaki nula.

$$\frac{\partial^2 u}{\partial x^2}=0,\quad \frac{\partial^2 v}{\partial x^2}=0,\quad \frac{\partial^2 u}{\partial y^2}=0,\quad \frac{\partial^2 v}{\partial y^2}=0$$

Po postavci je $\varphi=\varphi(u)$ и $\psi=\psi(v)$, sledi:

$$\frac{\partial \varphi}{\partial v}=0,\quad \frac{\partial \psi}{\partial u}=0 \tag{1}$$

a)

$$
\begin{aligned}
\frac{\partial z}{\partial x} &= \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial x}\\
&= \frac{\partial}{\partial u}\Big(\varphi(u)+\psi(v)\Big)\cdot 1+\frac{\partial}{\partial v}\Big(\varphi(u)+\psi(v)\Big)\cdot 1\\
&= \frac{\partial}{\partial u}\Big(\varphi(u)\Big)+\frac{\partial}{\partial u}\Big(\psi(v)\Big)+\frac{\partial}{\partial v}\Big(\varphi(u)\Big)+\frac{\partial}{\partial v}\Big(\psi(v)\Big)\\
&= \frac{\partial \varphi}{\partial u}+0+0+\frac{\partial \psi}{\partial v}\\
&= \frac{\partial \varphi}{\partial u}+\frac{\partial \psi}{\partial v}
\end{aligned}
$$

b) Slično kao pod a):

$$\begin{aligned}
\frac{\partial z}{\partial y} &= \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial y} \\
&= \frac{\partial}{\partial u}\Big(\varphi\left(u\right) + \psi\left(v\right)\Big) \cdot a + \frac{\partial}{\partial v}\Big(\varphi\left(u\right) + \psi\left(v\right)\Big) \cdot (-a) \\
&= a \cdot \left[\frac{\partial}{\partial u}\Big(\varphi\left(u\right)\Big) + \frac{\partial}{\partial u}\Big(\psi\left(v\right)\Big) - \frac{\partial}{\partial v}\Big(\varphi\left(u\right)\Big) - \frac{\partial}{\partial v}\Big(\psi\left(v\right)\Big)\right] \\
&= a \cdot \left(\frac{\partial \varphi}{\partial u} + 0 + 0 + \frac{\partial \psi}{\partial v}\right) \\
&= a \cdot \left(\frac{\partial \varphi}{\partial u} + \frac{\partial \psi}{\partial v}\right)
\end{aligned}$$

c)

$$\begin{aligned}
\frac{\partial^2 z}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{\partial \varphi}{\partial u} + \frac{\partial \psi}{\partial v}\right) \\
&= \frac{\partial}{\partial u}\left(\frac{\partial \varphi}{\partial u} + \frac{\partial \psi}{\partial v}\right) \cdot \frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial \varphi}{\partial u} + \frac{\partial \psi}{\partial v}\right) \cdot \frac{\partial v}{\partial x} \\
&= \left[\frac{\partial^2 \varphi}{\partial u^2} + \frac{\partial^2 \psi}{\partial u \partial v}\right] \cdot 1 + \left[\frac{\partial^2 \varphi}{\partial v \partial u} + \frac{\partial^2 \psi}{\partial v^2}\right] \cdot 1
\end{aligned}$$

Kako iz (1) sledi da je:

$$\frac{\partial^2 \varphi}{\partial v \partial u} = 0, \quad \frac{\partial^2 \psi}{\partial u \partial v} = 0$$

konačno se dobija:

$$\frac{\partial^2 z}{\partial x^2} = \left[\frac{\partial^2 \varphi}{\partial u^2} + 0\right] \cdot 1 + \left[0 + \frac{\partial^2 \psi}{\partial v^2}\right] \cdot 1 = \frac{\partial^2 \varphi}{\partial u^2} + \frac{\partial^2 \psi}{\partial v^2}$$

20. Neka je $r = x + y$ tada je:

$$\frac{\partial r}{\partial x} = 1, \quad \frac{\partial r}{\partial y} = 1 \tag{2}$$

dok data funkcija ima oblik:

$$u = f\left(r\right) + y \cdot g\left(r\right)$$

Nađimo:

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left(f\left(r\right)\right) + y \cdot \frac{\partial}{\partial x}\left(g\left(r\right)\right) = \frac{\partial f}{\partial r} \cdot \frac{\partial r}{\partial x} + y \cdot \frac{\partial g}{\partial r} \cdot \frac{\partial r}{\partial x} = \frac{\partial f}{\partial r} \cdot 1 + y \cdot \frac{\partial g}{\partial r} \cdot 1 = \frac{\partial f}{\partial r} + y \cdot \frac{\partial g}{\partial r}$$

slično se nalazi i izvod po y:

$$\begin{aligned}
\frac{\partial u}{\partial y} &= \frac{\partial}{\partial y}\left(f\left(r\right)\right)+g\left(r\right)+y\cdot\frac{\partial}{\partial y}\left(g\left(r\right)\right)\\
&= \frac{\partial f}{\partial r}\cdot\frac{\partial r}{\partial y}+g\left(r\right)+y\cdot\frac{\partial g}{\partial r}\cdot\frac{\partial r}{\partial y}\\
&= \frac{\partial f}{\partial r}\cdot 1+g\left(r\right)+y\cdot\frac{\partial g}{\partial r}\cdot 1\\
&= \frac{\partial f}{\partial r}+g\left(r\right)+y\cdot\frac{\partial g}{\partial r}
\end{aligned}$$

Da bi izračunali potrebnu vrednost nađimo druge izvode:

$$\frac{\partial^2 u}{\partial^2 x}=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial r}+y\cdot\frac{\partial g}{\partial r}\right)=\frac{\partial^2 f}{\partial^2 r}\cdot\frac{\partial r}{\partial x}+y\cdot\frac{\partial^2 g}{\partial^2 r}\cdot\frac{\partial r}{\partial x}=\frac{\partial^2 f}{\partial^2 r}\cdot 1+y\cdot\frac{\partial^2 g}{\partial^2 r}\cdot 1=\frac{\partial^2 f}{\partial^2 r}+y\cdot\frac{\partial^2 g}{\partial^2 r}$$

$$\begin{aligned}
\frac{\partial^2 u}{\partial x\partial y} &= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial r}+g\left(r\right)+y\cdot\frac{\partial g}{\partial r}\right)\\
&= \frac{\partial^2 f}{\partial^2 r}\cdot\frac{\partial r}{\partial x}+\frac{\partial g}{\partial r}\cdot\frac{\partial r}{\partial x}+y\cdot\frac{\partial^2 g}{\partial^2 r}\cdot\frac{\partial r}{\partial x}\\
&= \frac{\partial^2 f}{\partial^2 r}+\frac{\partial g}{\partial r}+y\cdot\frac{\partial^2 g}{\partial^2 r}
\end{aligned}$$

$$\frac{\partial^2 u}{\partial^2 y}=\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial r}+g\left(r\right)+y\cdot\frac{\partial g}{\partial r}\right)=\frac{\partial^2 f}{\partial^2 r}\cdot\frac{\partial r}{\partial y}+\cdot\frac{\partial g}{\partial r}\cdot\frac{\partial r}{\partial y}+\frac{\partial g}{\partial y}+y\cdot\frac{\partial^2 g}{\partial^2 r}\cdot\frac{\partial r}{\partial y}=\frac{\partial^2 f}{\partial^2 r}+2\cdot\frac{\partial g}{\partial r}+y\cdot\frac{\partial^2 g}{\partial^2 r}$$

$$\begin{aligned}
\frac{\partial^2 u}{\partial^2 x}-2\cdot\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial^2 u}{\partial^2 y} &= \frac{\partial^2 f}{\partial^2 r}+y\cdot\frac{\partial^2 g}{\partial^2 r}\\
&\quad -2\cdot\frac{\partial^2 f}{\partial^2 r}-2\cdot\frac{\partial g}{\partial r}-2y\cdot\frac{\partial^2 g}{\partial^2 r}\\
&\quad +\frac{\partial^2 f}{\partial^2 r}+2\cdot\frac{\partial g}{\partial r}+y\cdot\frac{\partial^2 g}{\partial^2 r}\\
&= 2\cdot\frac{\partial^2 f}{\partial^2 r}+2y\cdot\frac{\partial^2 g}{\partial^2 r}-2\cdot\frac{\partial^2 f}{\partial^2 r}-2\cdot\frac{\partial g}{\partial r}-2y\cdot\frac{\partial^2 g}{\partial^2 r}+2\cdot\frac{\partial g}{\partial r}\\
&= 0
\end{aligned}$$

21.

$$\nabla f=\left[\frac{2}{3}x^{-\frac{1}{3}},\quad \frac{2}{3}y^{-\frac{1}{3}},\quad \frac{2}{3}z^{-\frac{1}{3}}\right]$$

Tačka $M_0\left(x_0,y_0,z_0\right)$ pripada datoj površi:

$$x_0^{\frac{2}{3}}+y_0^{\frac{2}{3}}+z_0^{\frac{2}{3}}=4 \tag{3}$$

i vrednost gradijenta je:

$$\nabla\left(M_0\right)=\left[\frac{2}{3}x_0^{-\frac{1}{3}},\quad \frac{2}{3}y_0^{-\frac{1}{3}},\quad \frac{2}{3}z_0^{-\frac{1}{3}}\right]$$

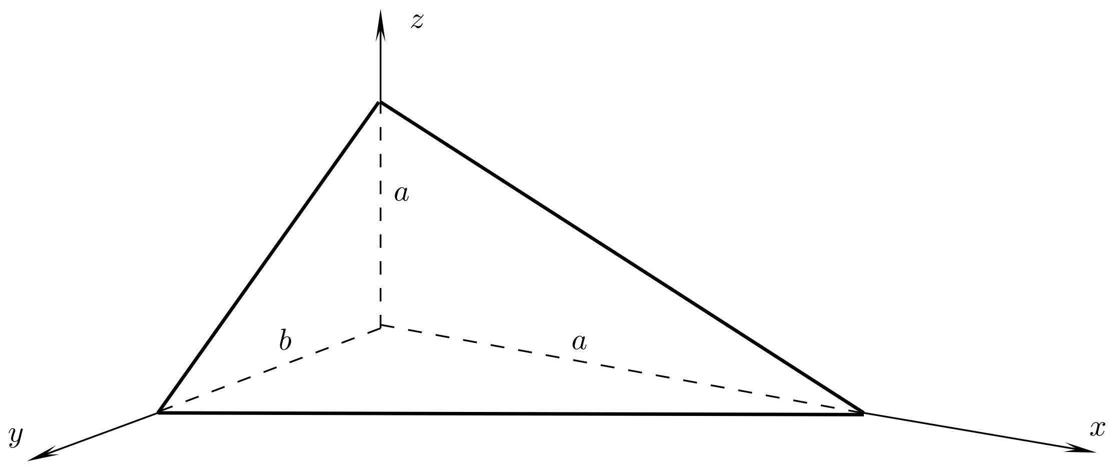
Formirajmo opšti oblik tangente ravni:

$$\Pi : \frac{2}{3}x_0^{-\frac{1}{3}}(X - x_0) + \frac{2}{3}y_0^{-\frac{1}{3}}(Y - y_0) + \frac{2}{3}z_0^{-\frac{1}{3}}(Z - z_0) = 0 \tag{4}$$

Transformišimo prethodnu jednačinu ravni u njen segmentni oblik:

$$\frac{X}{a} + \frac{Y}{b} + \frac{Z}{c} = 1$$

Na slici se vide potrebni odsečci.



$$\begin{aligned}
\Pi \;\; &\Rightarrow \;\; x_0^{-\frac{1}{3}}(X - x_0) + y_0^{-\frac{1}{3}}(Y - y_0) + z_0^{-\frac{1}{3}}(Z - z_0) = 0 \\
&\Rightarrow \;\; x_0^{-\frac{1}{3}}X + y_0^{-\frac{1}{3}}Y + z_0^{-\frac{1}{3}}Z = x_0^{\frac{2}{3}} + y_0^{\frac{2}{3}} + z_0^{\frac{2}{3}} \\
&\Rightarrow \;\; x_0^{-\frac{1}{3}}X + y_0^{-\frac{1}{3}}Y + z_0^{-\frac{1}{3}}Z = 4
\end{aligned}$$

jer važi (3) odakle je:

$$\Pi \;\; \Rightarrow \;\; \frac{X}{4x_0^{\frac{1}{3}}} + \frac{Y}{4y_0^{\frac{1}{3}}} + \frac{Z}{4z_0^{\frac{1}{3}}} = 1$$

Odsečke smo dobili, saberimo njihove kvadrate:

$$\left(4x_0^{\frac{1}{3}}\right)^2 + \left(4y_0^{\frac{1}{3}}\right)^2 + \left(4y_0^{\frac{1}{3}}\right)^2 = 16 \cdot \left(x_0^{\frac{2}{3}} + y_0^{\frac{2}{3}} + z_0^{\frac{1}{3}}\right) = 16 \cdot 4 = 4^3$$

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