M2; kol1; 17/18; &9; 23.3.2018

1 (4-1) [5]

$$I=\int\left(\frac{\sin x}{\cos^2 x}+x\cos x\right)dx=\overbrace{\int\frac{\sin x\,dx}{\cos^2 x}}^{=I_1}+\overbrace{\int x\cos x\,dx}^{=I_2}$$

$$I_1=\left|\begin{array}{l}\cos x=t\\ -\sin x\,dx=dt\end{array}\right| \; 1 \; =\int\frac{-dt}{t^2}=-\frac{t^{-1}}{-1}+C_1 \; 0.5 \; =\frac{1}{t}+C_1=\frac{1}{\cos x}+C_1 \; 0,5$$

$$I_2=\left|\begin{array}{l}u=x,\ dv=\cos x\,dx\\ du=dx,\ v=\sin x\end{array}\right| \; 1 \; =x\sin x-\int\sin x\,dx \; 0.5 \; =x\sin x-(-\cos x)+C_2$$

$$=x\sin x+\cos x+C_2 \quad 0.5$$

$$I=\frac{1}{\cos x}+x\sin x+\cos x\underbrace{+C}_{1}$$

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1 (4-2) [5]

$$I=\int\left(\frac{\cos x}{\sin^3 x}-x\sin x\right)dx=\overbrace{\int\frac{\cos x}{\sin^3 x}dx}^{=I_1}-\overbrace{\int x\sin x\,dx}^{=I_2}$$

$$I_1=\left|\begin{array}{l}\sin x=t\\ \cos x\,dx=dt\end{array}\right| \; 1 \; =\int\frac{dt}{t^3}=\frac{t^{-2}}{-2}+C_1=-\frac{1}{2t^2}+C_1 \; 0.5 \; =-\frac{1}{2\sin^2 x}+C_1 \; 0.5$$

$$I_2=\left|\begin{array}{l}u=x,\ dv=\sin x\,dx\\ du=dx,\ v=-\cos x\end{array}\right| \; 1 \; =-x\cos x+\int\cos x\,dx \; 0.5 \; =-x\cos x+\sin x+C_2 \quad 0.5$$

$$I=-\frac{1}{2\sin^2 x}+x\cos x-\sin x\underbrace{+C}_{1}$$

$\boxed{2}$ (T-1) $I=\int\frac{x^2+2x}{(x-1)^2(x^2+x+2)}\,dx$

[10]

$$\frac{x^2+2x}{(x-1)^2(x^2+x+2)}=\frac{A}{x-1}\,{}^{0.5}+\frac{B}{(x-1)^2}\,{}^{0.5}+\frac{Cx+D}{x^2+x+2}\,{}^{0.5}\qquad \Big|\,(x-1)^2(x^2+x+2)\quad 0.5$$

$$x^2+2x=A\underbrace{(x-1)(x^2+x+2)}_{\substack{=x^3+\cancel{x^2}+2x-\cancel{x^2}-x-2\\ =x^3+x-2}}+B(x^2+x+2)+(Cx+D)\underbrace{(x-1)^2}_{x^2-2x+1}$$

$$=Ax^3+Ax-2A+Bx^2+Bx+2B+Cx^3-2Cx^2+Cx+Dx^2-2Dx+D$$

x^3: $0=A+C\Rightarrow C=-A$

x^2: $1=B-2C+D$ (2)

x: $2=A+B+C-2D$

1: $0=-2A+2B+D$

$$2A+B+D=1$$
$$B-2D=2$$
$$-2A+2B+D=0$$

$(+)$ $3B+2D=1$, $B-2D=2$ $(+)$

$4B=3\Rightarrow B=3/4$

$D=\frac{1}{2}(B-2)=-5/8$

$A=\frac{1}{2}(2B+D)=7/16$ (1)

$C=-7/16$

$$I=\int\left(\frac{7/16}{x-1}+\frac{3/4}{(x-1)^2}+\frac{\overbrace{-7/16\,x-5/8}^{-\frac{7}{32}(2x+1)+\frac{7}{32}-\frac{5}{8}\ (1)}}{x^2+x+2}\right)dx$$

$$=\frac{7}{16}\int\frac{dx}{x-1}+\frac{3}{4}\int\frac{dx}{(x-1)^2}-\frac{7}{32}\int\frac{2x+1}{x^2+x+2}dx-\frac{13}{32}\underbrace{\int\frac{dx}{x^2+x+2}}_{I_1}$$

$$x^2+x+2=\left(x+\frac{1}{2}\right)^2+\frac{7}{4}=\frac{7}{4}\left(\left(\frac{\cancel{2}}{\sqrt7}\frac{2x+1}{\cancel{2}}\right)^2+1\right)\quad (1)$$

$$I_1=\int\frac{dx}{\frac{7}{4}\left(\left(\frac{2x+1}{\sqrt7}\right)^2+1\right)}=\left|\begin{array}{l}\frac{2x+1}{\sqrt7}=t\quad 0.5\\ \frac{2}{\sqrt7}dx=dt\end{array}\right|=\int\frac{\frac{\sqrt7}{2}dt}{\frac{7}{4}(t^2+1)}=\frac{2}{\sqrt7}\operatorname{arctg}t+C$$

$$=\frac{2}{\sqrt7}\operatorname{arctg}\frac{2x+1}{\sqrt7}+C$$

$$I=\underset{0.5}{\frac{7}{16}\ln|x-1|}-\underset{0.5}{\frac{3}{4(x-1)}}-\underset{0.5}{\frac{7}{32}\ln(x^2+x+2)}-\underset{0.5}{\frac{13}{16\sqrt7}\operatorname{arctg}\frac{2x+1}{\sqrt7}}+\underset{0.5}{C}$$

$\boxed{2}$ (4.2) $\displaystyle\int \frac{x^2-2x}{(x+1)^2(x^2-x+2)}\,dx$

[10]

$$\frac{x^2-2x}{(x+1)^2(x^2-x+2)} = \frac{A}{x+1} \;(0.5) + \frac{B}{(x+1)^2} \;(0.5) + \frac{Cx+D}{x^2-x+2} \;(0.5) \quad \Big/ (x+1)^2(x^2-x+2) \quad (0.5)$$

$$x^2-2x = A\underbrace{(x+1)(x^2-x+2)}_{\substack{=x^3\cancel{-x^2}+2x\cancel{+x^2}-x+2 \\ =x^3+x+2}} + B(x^2-x+2) + (Cx+D)\underbrace{(x+1)^2}_{x^2+2x+1}$$

$$= Ax^3 + Ax + 2A + Bx^2 - Bx + 2B + Cx^3 + 2Cx^2 + Cx + Dx^2 + 2Dx + D$$

$$\begin{aligned}
x^3 &: \; 0 = A + C \Rightarrow C = -A \\
x^2 &: \; 1 = B + 2C + D \quad (2) \\
x &: \; -2 = A - B + C + 2D \\
1 &: \; 0 = 2A + 2B + D
\end{aligned}
\qquad
\begin{aligned}
-2A + B + D &= 1 \\
-B + 2D &= -2 \\
2A + 2B + D &= 0
\end{aligned}
\quad (+)
\qquad
\begin{aligned}
3B + 2D &= 1 \\
-B + 2D &= -2 \quad / \cdot (-1) \;(+) \\
\hline
4B = 3 &\Rightarrow B = 3/4 \\
D = \tfrac{1}{2}(B-2) &= -5/8 \\
A = -\tfrac{1}{2}(2B+D) &= -7/16 \\
C &= 7/16
\end{aligned}$$

(1)

$$I = \int\left(\frac{-7/16}{x+1} + \frac{3/4}{(x+1)^2} + \frac{\overbrace{7/16\,x - 5/8}^{\frac{7}{32}(2x-1)+\frac{7}{32}-\frac{5}{8}\;(1)}}{x^2-x+2}\right)dx$$

$$= -\frac{7}{16}\int\frac{dx}{x+1} + \frac{3}{4}\int\frac{dx}{(x+1)^2} + \frac{7}{32}\int\frac{2x-1}{x^2-x+2}\,dx - \frac{13}{32}\underbrace{\int\frac{dx}{x^2-x+2}}_{I_1}$$

$$x^2-x+2 = \left(x-\frac{1}{2}\right)^2 + \frac{7}{4} = \frac{7}{4}\left(\left(\frac{\cancel{2}}{\sqrt7}\,\frac{2x-1}{\cancel{2}}\right)^2+1\right) \quad (1)$$

$$I_1 = \int\frac{dx}{\frac{\cancel{7}}{\cancel{4}}\left(\left(\frac{2x-1}{\sqrt7}\right)^2+1\right)} = \left[\begin{aligned}\frac{2x-1}{\sqrt7} &= t \\ \frac{2dx}{\sqrt7} &= dt\end{aligned}\;(0.5)\right] = \int\frac{\frac{\sqrt7}{2}\,dt}{\frac{\cancel{7}}{\cancel{4}}(t^2+1)} = \frac{2}{\sqrt7}\,\mathrm{arctg}\,t + C$$

$$= \frac{2}{\sqrt7}\,\mathrm{arctg}\,\frac{2x-1}{\sqrt7} + C$$

$$I = -\frac{7}{16}\ln|x+1| - \frac{3}{4(x+1)} + \frac{7}{32}\ln(x^2-x+2) - \frac{13}{16\sqrt7}\,\mathrm{arctg}\,\frac{2x-1}{\sqrt7} + C$$

(0.5) (0.5) (0.5) (0.5) (0.5)

3. (4-1) [10]

$$I=\int \frac{x\ln^2\sqrt[6]{x}}{\sqrt[6]{x^5}}\,dx=\left\{\begin{aligned}&x^{1/6}=t\\&\tfrac{1}{6}x^{-5/6}dx=dt\end{aligned}\right\}\;\text{2}\;=6\int t^6\ln^2 t\,dt=\;\text{0.5}$$

$$=\left\{\begin{aligned}&u=\ln^2 t,\ dv=t^6dt\\&du=2\ln t\cdot\tfrac{1}{t}dt,\ v=\tfrac{t^7}{7}\end{aligned}\right\}\;\text{2}\;=6\cdot\left(\frac{t^7\ln^2 t}{7}\;\text{1}\;-\frac{2}{7}\int t^6\ln t\,dt\right)=$$

$$=\left\{\begin{aligned}&u=\ln t,\ dv=t^6dt\\&du=\tfrac{dt}{t},\ v=\tfrac{t^7}{7}\end{aligned}\right\}\;\text{2}\;=\frac{6t^7\ln^2 t}{7}-\frac{12}{7}\left(\frac{t^7\ln t}{7}\;\text{1}\;-\frac{1}{7}\int t^6dt\right)=$$

$$=\frac{6t^7\ln^2 t}{7}-\frac{12t^7\ln t}{49}-\frac{12}{49}\cdot\frac{t^7}{7}\;\text{0.5}\;+C\;\text{0.5}\;=$$

$$=\frac{6t^7}{343}\left(49\ln^2 t-14\ln t-2\right)+C=$$

$$=\frac{6\sqrt[6]{x^7}}{343}\left(49\ln^2\sqrt[6]{x^7}-14\ln\sqrt[6]{x^7}-2\right)+C=\;\text{0.5}$$

3. (4-2) [10]

$$I=\int \frac{x\ln^2\sqrt[5]{x}}{\sqrt[5]{x^4}}\,dx=\left\{\begin{aligned}&x^{1/5}=t\\&\tfrac{1}{5}x^{-4/5}dx=dt\end{aligned}\right\}\;\text{2}\;=5\int t^5\ln^2 t\,dt=\;\text{0.5}$$

$$=\left\{\begin{aligned}&u=\ln^2 t,\ dv=t^5dt\\&du=2\ln t\cdot\tfrac{1}{t}dt,\ v=\tfrac{t^6}{6}\end{aligned}\right\}\;\text{2}\;=5\left(\frac{t^6\ln^2 t}{6}\;\text{1}\;-\frac{1}{3}\int t^5\ln t\,dt\right)=$$

$$=\left\{\begin{aligned}&u=\ln t,\ dv=t^5dt\\&du=\tfrac{dt}{t},\ v=\tfrac{t^6}{6}\end{aligned}\right\}\;\text{2}\;=\frac{5t^6\ln^2 t}{6}-\frac{5}{3}\left(\frac{t^6\ln t}{6}\;\text{1}\;-\frac{1}{6}\int t^5dt\right)=$$

$$=\frac{5t^6\ln^2 t}{6}-\frac{5t^6\ln t}{18}-\frac{5}{18}\cdot\frac{t^6}{6}\;\text{0.5}\;+C\;\text{0.5}\;=$$

$$=\frac{5t^6}{108}\left(18\ln^2 t-6\ln t-1\right)+C=$$

$$=\frac{5\sqrt[5]{x^6}}{108}\left(18\ln^2\sqrt[5]{x^6}-6\ln\sqrt[5]{x^6}-1\right)+C\;\text{0.5}$$

$\boxed{4.}$ (Пр-1) $I=\int \frac{dx}{(x+1)\sqrt{-x^2+5x-4}}$

$\boxed{10}$

$-x^2+5x-4=-(x-1)(x-4)$ 1

Ојлерова смена:

$-(x-1)=(x-4)t^2$

$-x+1=xt^2-4t^2$

$xt^2+x=4t^2+1$

$x(t^2+1)=4t^2+1$

$x=\frac{4t^2+1}{t^2+1}$ 2

$dx=\frac{8t(t^2+1)-(4t^2+1)\cdot 2t}{(t^2+1)^2}dt$

$=\frac{\cancel{8t^3}+8t-\cancel{8t^3}-2t}{(t^2+1)^2}dt$

$=\frac{6t\,dt}{(t^2+1)^2}$ 2

$x+1=\frac{4t^2+1}{t^2+1}+\frac{t^2+1}{t^2+1}=\frac{5t^2+2}{t^2+1}$ 1

$\sqrt{-x^2+5x-4}=\sqrt{-(x-1)(x-4)}=\sqrt{(x-4)t^2(x-4)}=(x-4)t=$

$=\left(\frac{4t^2+1}{t^2+1}-\frac{4t^2+4}{t^2+1}\right)t=-\frac{3t}{t^2+1}$ 1

$I=\int\frac{\frac{\overset{2}{\cancel{6t}}\,dt}{\cancel{(t^2+1)^2}}}{\frac{5t^2+2}{\cancel{t^2+1}}\left(-\frac{\cancel{3t}}{\cancel{t^2+1}}\right)}=-\int\frac{dt}{\frac{5t^2}{2}+1}=\left[\begin{array}{l}t\sqrt{\frac{5}{2}}=s\\ \sqrt{\frac{5}{2}}\,dt=ds\end{array}\right]=$ 1 0.5

$=-\int\frac{\sqrt{\frac{2}{5}}\,ds}{s^2+1}=-\sqrt{\frac{2}{5}}\operatorname{arctg}s+C=-\sqrt{\frac{2}{5}}\operatorname{arctg}t\sqrt{\frac{5}{2}}+C$ 0.5

$=-\sqrt{\frac{2}{5}}\operatorname{arctg}\sqrt{\frac{5(1-x)}{2(x-4)}}+C$

0.5 0.5

$\boxed{4}$ (4.2) $I = \int \frac{dx}{(x-1)\sqrt{-x^2+5x+6}}$

$-x^2+5x-6 = -(x-2)(x-3)$

Ојлерова смена:

$-(x-2) = (x-3)t^2$

$-x+2 = xt^2 - 3t^2$

$xt^2 + x = 3t^2+2$

$x = \frac{3t^2+2}{t^2+1}$

$dx = \frac{6t(t^2+1)-(3t^2+2)\cdot 2t}{(t^2+1)^2}\,dt$

$= \frac{\cancel{6t^3}+6t-\cancel{6t^3}-4t}{(t^2+1)^2}\,dt$

$= \frac{2t\,dt}{(t^2+1)^2}$

$x-1 = \frac{3t^2+2}{t^2+1} - \frac{t^2+1}{t^2+1} = \frac{2t^2+1}{t^2+1}$

$\sqrt{-x^2+5x-6} = \sqrt{-(x-2)(x-3)} = \sqrt{(x-3)t^2(x-3)} = (x-3)t =$

$= \left(\frac{3t^2+2}{t^2+1} - \frac{3t^2+3}{t^2+1}\right)t = \frac{-t}{t^2+1}$

$I = \int \frac{\frac{2t\,dt}{\cancel{(t^2+1)^2}}}{\frac{2t^2+1}{\cancel{t^2+1}}\cdot\frac{-t}{\cancel{t^2+1}}} = -2\int \frac{dt}{2t^2+1} = \left\{ \begin{matrix} t\sqrt{2} = s \\ \sqrt{2}\,dt = ds \end{matrix} \right\} =$

$= -2\int \frac{\frac{ds}{\sqrt{2}}}{s^2+1} = -\sqrt{2}\,\mathrm{arctg}\, s + C = -\sqrt{2}\,\mathrm{arctg}\, t\sqrt{2} + C =$

$= -\sqrt{2}\,\mathrm{arctg}\sqrt{\frac{2(2-x)}{x-3}} + C$