Мат. 2; кол. 2; 17/18; 89 (14.5.2018.)

1. [5]

$$\int_0^{\infty} x e^{-ax}\,dx \overset{1}{=} \lim_{\beta\to\infty}\int_0^{\beta} x e^{-ax}\,dx = \left\{ \begin{matrix} u = x, & dv = e^{-ax}dx \\ du = dx, & v = -\frac{1}{a}e^{-ax} \end{matrix} \right\} \overset{1}{=}$$

$(a \neq 0)$

$$= \lim_{\beta\to\infty}\left(\left[-\frac{1}{a} x e^{-ax}\right]\Big|_0^{\beta} + \frac{1}{a}\int_0^{\beta} e^{-ax}\,dx \right) =$$

$$\underset{\beta\to\infty}{\lim}\left(-\frac{1}{a}\left(\beta e^{-a\beta} - 0\cdot 1\right) + \frac{1}{a}\left(-\frac{1}{a}e^{-ax}\right)\Big|_0^{\beta}\right)$$

замена границе: 1

остатак рачуна: 1

$$= \lim_{\beta\to\infty}\left(-\frac{1}{a}\frac{\beta}{e^{a\beta}} - \frac{1}{a^2}\left(e^{-a\beta} - 1\right)\right) = \lim_{\beta\to\infty}\left(-\frac{1}{a}\cancel{\frac{\beta}{e^{a\beta}}}^{0} - \frac{1}{a^2}\cancel{\frac{1}{e^{a\beta}}}^{0} + \frac{1}{a^2}\right)$$

$$= \frac{1}{a^2}$$

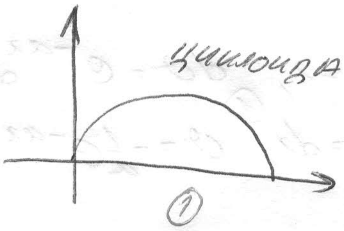
$$\lim_{\beta\to\infty}\frac{\beta}{e^{a\beta}} \overset{\text{Л.}\frac{\infty}{\infty}}{=} \lim_{\beta\to\infty}\frac{1}{a e^{a\beta}} = 0 \quad 1$$

(гр. 1) $a = 2$, реш. $\frac{1}{4}$

(гр. 2) $a = 3$ реш. $\frac{1}{9}$

[2] $x = a(t-\sin t)$

[10] $y = a(1-\cos t)$ $\quad t\in[0,\ k\pi/2]\quad (0\le k\le 4)$



①

$x' = a(1-\cos t)$ ①

$y' = a\sin t$ ①

$x'^2+y'^2 = a^2(1-2\cos t+\underbrace{\cos^2 t+\sin^2 t}_{=1})$

$= 2a^2(1-\cos t)$ ①

ф-ула: ①+①

$$L=\int_0^{k\pi/2}\sqrt{2a^2\boxed{(1-\cos t)}}\,dt \overset{①}{=} a\sqrt2\int_0^{k\pi/2}\sqrt{\boxed{2\sin^2\frac t2}}\,dt$$

$$= 2a\int_0^{k\pi/2}\overset{①}{\left|\sin\frac t2\right|}dt = 2a\int_0^{k\pi/2}\overset{①}{\sin\frac t2}\,dt = 2a\left[-2\cos\frac t2\right]\Big|_0^{k\pi/2} =$$

$$= -4a\left(\cos\frac{k\pi}{4}-1\right)$$ ①

(Пр.1) $a=3,\ k=1$ $\quad$ реш. $-12\cancel{a}\left(\frac{1}{\sqrt2}-1\right) = \frac{12\cancel{a}(\sqrt2-1)}{\sqrt2}$

$= 6\cancel{a}\sqrt2(\sqrt2-1)$

(Пр.2) $a=2,\ k=3$ $\quad$ реш. $-8\cancel{a}\left(-\frac{1}{\sqrt2}-1\right) = \frac{8\cancel{a}(\sqrt2+1)}{\sqrt2}$

$= 4\cancel{a}\sqrt2(\sqrt2+1)$

$\boxed{3.}$ $$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy \quad (1)$$

[10] + [5]

$$d^2z = \frac{\partial^2 z}{\partial x^2}dx^2 + 2\frac{\partial^2 z}{\partial x \partial y}dx\,dy + \frac{\partial^2 z}{\partial y^2}dy^2 \quad (1)$$

$$d^3z = \frac{\partial^3 z}{\partial x^3}dx^3 + 3\frac{\partial^3 z}{\partial x^2 \partial y}dx^2dy + 3\frac{\partial^3 z}{\partial x \partial y^2}dx\,dy^2 + \frac{\partial^3 z}{\partial y^3}dy^3 \quad (2)$$

(4.1) $$z = \arcsin\sqrt{\frac{x^2-y^2}{x^2+y^2}}$$

$$\frac{\partial z}{\partial x} = \frac{1}{\sqrt{1-\frac{x^2-y^2}{x^2+y^2}}} \cdot \frac{1}{\cancel{2}\sqrt{\frac{x^2-y^2}{x^2+y^2}}} \cdot \frac{\cancel{2}x(x^2+y^2) - (x^2-y^2)\cdot \cancel{2}x}{(x^2+y^2)^2}$$

$$= \frac{1}{\sqrt{\frac{\cancel{x^2}+y^2-\cancel{x^2}+y^2}{\cancel{x^2+y^2}}}}\sqrt{\frac{\cancel{x^2+y^2}}{x^2-y^2}} \cdot \frac{\cancel{x^3}+xy^2-\cancel{x^3}+xy^2}{(x^2+y^2)^{\cancel{2}}}$$

$$= \frac{1}{\cancel{y}\sqrt{2}} \cdot \frac{2xy^{\cancel{2}}}{x^2+y^2} \cdot \frac{1}{\sqrt{x^2-y^2}} = \frac{xy\sqrt{2}}{(x^2+y^2)\sqrt{x^2-y^2}} \quad (1)$$

$$\frac{\partial z}{\partial y} = \frac{1}{\sqrt{1-\frac{x^2-y^2}{x^2+y^2}}} \cdot \frac{1}{\cancel{2}\sqrt{\frac{x^2-y^2}{x^2+y^2}}} \cdot \frac{-\cancel{2}y(x^2+y^2) - (x^2-y^2)\cdot \cancel{2}y}{(x^2+y^2)^2}$$

$$= \frac{1}{\sqrt{\frac{\cancel{x^2}+y^2-\cancel{x^2}+y^2}{\cancel{x^2+y^2}}}}\sqrt{\frac{\cancel{x^2+y^2}}{x^2-y^2}} \cdot \frac{-x^2y-\cancel{y^3}-x^2y+\cancel{y^3}}{(x^2+y^2)^{\cancel{2}}}$$

$$= \frac{1}{\cancel{y}\sqrt{2}} \cdot \frac{1}{\sqrt{x^2-y^2}} \cdot \frac{-2x^2\cancel{y}}{x^2+y^2} = \frac{-x^2\sqrt{2}}{(x^2+y^2)\sqrt{x^2-y^2}} \quad (1)$$

$$\frac{\partial z}{\partial x}=\frac{xy\sqrt{2}}{(x^2+y^2)\sqrt{x^2-y^2}} \qquad \frac{\partial z}{\partial y}=\frac{-x^2\sqrt{2}}{(x^2+y^2)\sqrt{x^2-y^2}} \quad \left(=-x^2\sqrt{2}\,(x^2+y^2)^{-1}(x^2-y^2)^{-\frac{1}{2}}\right)$$

$$\frac{\partial^2 z}{\partial x^2}=y\sqrt{2}\,\frac{(x^2+y^2)\sqrt{x^2-y^2}-x\cdot\left(2x\sqrt{x^2-y^2}+(x^2+y^2)\cdot\frac{\cancel{2}x}{\cancel{2}\sqrt{x^2-y^2}}\right)}{(x^2+y^2)^2(x^2-y^2)}$$

$$=y\sqrt{2}\,\frac{\frac{(x^2+y^2)(x^2-y^2)-2x^2(x^2-y^2)-x^2(x^2+y^2)}{\sqrt{x^2-y^2}}}{(x^2+y^2)^2(x^2-y^2)}$$

$$=y\sqrt{2}\,\frac{\cancel{x^4}-y^4-2x^4+2x^2y^2-\cancel{x^4}-x^2y^2}{(x^2+y^2)^2(x^2-y^2)^{3/2}}$$

$$=y\sqrt{2}\,\frac{x^2y^2-2x^4-y^4}{(x^2+y^2)^2(x^2-y^2)^{3/2}} \quad (2)$$

$$\frac{\partial^2 z}{\partial x\partial y}=x\sqrt{2}\,\frac{(x^2+y^2)\sqrt{x^2-y^2}-y\left(2y\sqrt{x^2-y^2}+(x^2+y^2)\frac{-\cancel{2}y}{\cancel{2}\sqrt{x^2-y^2}}\right)}{(x^2+y^2)^2(x^2-y^2)}$$

$$=x\sqrt{2}\,\frac{\frac{(x^2+y^2)(x^2-y^2)-2y^2(x^2-y^2)-y^2(x^2+y^2)}{\sqrt{x^2-y^2}}}{(x^2+y^2)^2(x^2-y^2)}$$

$$=x\sqrt{2}\,\frac{x^4\cancel{-y^4}-2x^2y^2\cancel{+y^4}-x^2y^2-y^4}{(x^2+y^2)^2(x^2-y^2)^{3/2}}$$

$$=x\sqrt{2}\,\frac{x^4-y^4-3x^2y^2}{(x^2+y^2)^2(x^2-y^2)^{3/2}} \quad (2)$$

$$\frac{\partial^2 z}{\partial y^2} = -x^2\sqrt{2}\left(-(x^2+y^2)^{-2}\cdot 2y\,(x^2-y^2)^{-\frac{1}{2}} + (x^2+y^2)^{-1}\left(-\frac{1}{2}\right)(x^2-y^2)^{-\frac{3}{2}}(-2y)\right)$$

$$= -x^2\sqrt{2}\left(\frac{-2y}{(x^2+y^2)^2\sqrt{x^2-y^2}} + \frac{y}{(x^2+y^2)(x^2-y^2)^{3/2}}\right)$$

$$= -x^2y\sqrt{2}\,\frac{-2(x^2-y^2)+(x^2+y^2)}{(x^2+y^2)^2(x^2-y^2)^{3/2}}$$

$$= -x^2y\sqrt{2}\,\frac{-2x^2+2y^2+x^2+y^2}{(x^2+y^2)^2(x^2-y^2)^{3/2}} = x^2y\sqrt{2}\,\frac{x^2-3y^2}{(x^2+y^2)^2(x^2-y^2)^{3/2}}$$

(2)

Бодови (10): ф-ја; $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$; 2. извода (дело шта)

Питање (5): до 1: 2. извод (прорачун), 3. извода

(Z2) $z = \arccos\sqrt{\frac{y^2-x^2}{x^2+y^2}}$

$$\frac{\partial z}{\partial x} = -\frac{1}{\sqrt{1-\frac{y^2-x^2}{x^2+y^2}}}\cdot\frac{1}{2\sqrt{\frac{y^2-x^2}{x^2+y^2}}}\cdot\frac{-2x(x^2+y^2)-(y^2-x^2)\cdot 2x}{(x^2+y^2)^2}$$

$$= -\frac{1}{\sqrt{\frac{x^2+y^2-y^2+x^2}{x^2+y^2}}}\sqrt{\frac{x^2+y^2}{y^2-x^2}}\;\frac{-x^3-xy^2-xy^2+x^3}{(x^2+y^2)^2}$$

$$= -\frac{1}{x\sqrt{2}}\;\frac{1}{\sqrt{y^2-x^2}}\;\frac{-2xy^2}{x^2+y^2} = \frac{y^2\sqrt{2}}{(x^2+y^2)\sqrt{y^2-x^2}}$$

$$\frac{\partial z}{\partial y} = -\frac{1}{\sqrt{1-\frac{y^2-x^2}{x^2+y^2}}}\cdot\frac{1}{2\sqrt{\frac{y^2-x^2}{x^2+y^2}}}\cdot\frac{2y(x^2+y^2)-(y^2-x^2)\cdot 2y}{(x^2+y^2)^2} =$$

$$= -\frac{1}{\sqrt{\frac{x^2+y^2-y^2+x^2}{x^2+y^2}}}\sqrt{\frac{x^2+y^2}{y^2-x^2}}\;\frac{x^2y+y^3-y^3+x^2y}{(x^2+y^2)^2}$$

$$= -\frac{1}{x\sqrt{2}}\cdot\frac{1}{\sqrt{y^2-x^2}}\,\frac{2x^2y}{x^2+y^2} = -\frac{xy\sqrt{2}}{(x^2+y^2)\sqrt{y^2-x^2}}$$

$$\frac{\partial z}{\partial x} = \frac{y^2\sqrt{2}}{(x^2+y^2)\sqrt{y^2-x^2}}, \quad \frac{\partial z}{\partial y} = -\frac{xy\sqrt{2}}{(x^2+y^2)\sqrt{y^2-x^2}}$$

$$\frac{\partial^2 z}{\partial y^2} = -x\sqrt{2}\,\frac{x^2y^2-x^4-2y^4}{(x^2+y^2)^2(y^2-x^2)^{3/2}}$$

$$\frac{\partial^2 z}{\partial x\partial y} = -y\sqrt{2}\,\frac{y^4-x^4-3x^2y^2}{(x^2+y^2)^2(y^2-x^2)^{3/2}}$$

$$\frac{\partial^2 z}{\partial x^2} = -xy^2\sqrt{2}\,\frac{y^2-3x^2}{(x^2+y^2)^2(y^2-x^2)^{3/2}}$$

[4] (вар.1) $z = xy + x\varphi\left(\frac{x}{y}\right)$

[10]

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = xy + z$$

$$\frac{x}{y} = t; \quad z = xy + x\varphi(t); \quad \frac{\partial t}{\partial x} = \frac{1}{y}, \quad \frac{\partial t}{\partial y} = \boxed{-\frac{x}{y^2}}$$

$$\frac{\partial z}{\partial x} = y + \varphi(t) + x\frac{\partial \varphi}{\partial x} = y + \varphi\left(\frac{x}{y}\right) + x\cdot\frac{\partial \varphi}{\partial t}\cdot\frac{\partial t}{\partial x}$$

$$= y + \varphi\left(\frac{x}{y}\right) + \frac{x}{y}\frac{d\varphi}{dt}$$

$$\frac{\partial z}{\partial y} = x + x\frac{\partial \varphi}{\partial y} = x + x\frac{\partial \varphi}{\partial t}\boxed{\frac{\partial t}{\partial y}} = x - \frac{x^2}{y^2}\frac{d\varphi}{dt}$$

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = \underbrace{xy + x\varphi\left(\frac{x}{y}\right)} + \cancel{\frac{x^2}{y}\frac{d\varphi}{dt}} + xy - \cancel{\frac{x^2}{y}\frac{d\varphi}{dt}}$$

$$= xy + \underbrace{z}.$$

[4] (вар.2) $z = xy + y\varphi\left(\frac{y}{x}\right)$

[10]

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = xy + z$$

$$\frac{y}{x} = t, \quad z = xy + y\varphi(t); \quad \frac{\partial t}{\partial x} = -\frac{y}{x^2}, \quad \frac{\partial t}{\partial y} = \boxed{\frac{1}{x}}$$

$$\frac{\partial z}{\partial x} = y + y\cdot\frac{\partial \varphi}{\partial x} = y + y\cdot\frac{\partial \varphi}{\partial t}\cdot\frac{\partial t}{\partial x} = y - \frac{y^2}{x^2}\frac{d\varphi}{dt}$$

$$\frac{\partial z}{\partial y} = x + \varphi(t) + y\frac{\partial \varphi}{\partial y} = x + \varphi\left(\frac{y}{x}\right) + y\frac{\partial \varphi}{\partial t}\boxed{\frac{\partial t}{\partial y}}$$

$$= x + \varphi\left(\frac{y}{x}\right) + \frac{y}{x}\frac{d\varphi}{dt}$$

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = xy - \cancel{\frac{y^2}{x}\frac{d\varphi}{dt}} + \underbrace{xy + y\varphi\left(\frac{y}{x}\right)} + \cancel{\frac{y^2}{x}\frac{d\varphi}{dt}}$$

$$= xy + \underbrace{z}.$$