Rešenja

1. grupa

1. Rotor polja $\vec{A}$ je

$$\begin{aligned}
\mathrm{rot}\vec{A} = \nabla \times \vec{A} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ xz & xy & yz \end{vmatrix} \\
&= \left(\frac{\partial}{\partial y}(yz) - \frac{\partial}{\partial z}(xy)\right)\vec{i} - \left(\frac{\partial}{\partial x}(yz) - \frac{\partial}{\partial z}(xz)\right)\vec{j} + \left(\frac{\partial}{\partial x}(xy) - \frac{\partial}{\partial y}(xz)\right)\vec{k} \\
&= (z-0)\,\vec{i} - (0-x)\,\vec{j} + (y-0)\,\vec{k} = z\,\vec{i} + x\,\vec{j} + y\,\vec{k}.
\end{aligned}$$

Rotor polja $\vec{A}$ u tački X je

$$\mathrm{rot}\vec{A}(X) = \vec{i} + \vec{j},$$

a njegov intenzitet iznosi

$$|\mathrm{rot}\vec{A}(X)| = \sqrt{1^2 + 1^2} = \sqrt{2}.$$

2. Kriva C je kružnica koja se nalazi u preseku sfere $x^2 + y^2 + z^2 = 1$ i ravni $z = -\frac{1}{2}$. Ako $z = -\frac{1}{2}$ uvrstimo u jednačinu sfere dobijamo

$$x^2 + y^2 = \frac{3}{4},$$

pa je parametrizacija kružnice C

$$x = \frac{\sqrt{3}}{2}\cos t,\ y = \frac{\sqrt{3}}{2}\sin t,\ z = -\frac{1}{2};\quad 0 \le t \le 2\pi.$$

Odavde je

$$ds = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}dt = \sqrt{\left(-\frac{\sqrt{3}}{2}\sin t\right)^2 + \left(\frac{\sqrt{3}}{2}\cos t\right)^2 + 0^2}dt = \frac{\sqrt{3}}{2}dt$$

i

$$\begin{aligned}
\oint_C y^2 ds &= \int_0^{2\pi} \left(\frac{\sqrt{3}}{2}\sin t\right)^2 \frac{\sqrt{3}}{2}dt = \frac{3\sqrt{3}}{8}\int_0^{2\pi} \sin^2 t dt = \frac{3\sqrt{3}}{8}\int_0^{2\pi} \frac{1-\cos 2t}{2}dt \\
&= \frac{3\sqrt{3}}{16}\left(t - \frac{1}{2}\sin 2t\right)\Big|_0^{2\pi} = \frac{3\pi\sqrt{3}}{8}.
\end{aligned}$$

3. Radi se slično kao 3. zadatak 2. grupe, s tim što je ovde x zamenjeno sa y i obrnuto, y sa x. Rezultat je isti.

2. grupa

1. Rotor polja $\vec{A}$ je

$$\text{rot}\vec{A} = \nabla \times \vec{A} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz & xy & yz \end{vmatrix}$$

$$= \left(\frac{\partial}{\partial y}(yz) - \frac{\partial}{\partial z}(xy)\right)\vec{i} - \left(\frac{\partial}{\partial x}(yz) - \frac{\partial}{\partial z}(xz)\right)\vec{j} + \left(\frac{\partial}{\partial x}(xy) - \frac{\partial}{\partial y}(xz)\right)\vec{k}$$

$$= (z-0)\,\vec{i} - (0-x)\,\vec{j} + (y-0)\,\vec{k} = z\,\vec{i} + x\,\vec{j} + y\,\vec{k}.$$

Rotor polja $\vec{A}$ u tački X je

$$\text{rot}\vec{A}(X) = 2\vec{i} + \vec{j},$$

a njegov intenzitet iznosi

$$|\text{rot}\vec{A}(X)| = \sqrt{2^2 + 1^2} = \sqrt{5}.$$

2. Zadatak se rešava kao 2. zadatak za 1. grupu, ovde je x zamenjeno sa y, y sa z, i z sa x. Rezultat je isti.

3. Radi se o delu površi $z(x,y) = \dfrac{x^2+y^2}{4}$ koji se jednoznačno projektuje u oblast

$$G = \left\{(x,y) | (x^2+y^2)^2 \leq 4(x^2-y^2)\right\}.$$

Tražena površina iznosi

$$S = \iint\limits_G \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dx\, dy = \iint\limits_G \sqrt{1 + \frac{x^2}{4} + \frac{y^2}{4}}\, dx\, dy.$$

Za $x = \rho\cos\varphi$ i $y = \rho\sin\varphi$ važi

$$(x^2+y^2)^2 \leq 4(x^2-y^2) \Leftrightarrow \rho^4 \leq 4\rho^2\left(\cos^2\varphi - \sin^2\varphi\right) = 4\rho^2\cos 2\varphi \Leftrightarrow \rho \leq 2\sqrt{\cos 2\varphi},$$

pa opis oblasti G u polarnim koordinatama glasi

$$G' = \left\{(\rho,\varphi) | \, 0 \leq \rho \leq \sqrt{2\cos\varphi},\ -\frac{\pi}{4} \leq \varphi \leq \frac{\pi}{4}\right\}$$

(granice za φ su proistekle iz uslova $x \geq 0$ i $x^2 \geq y^2$). Dakle,

$$\begin{aligned} S &= \iint\limits_{G'} \sqrt{1+\frac{\rho^2}{4}}\,\rho\, d\rho\, d\varphi = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} d\varphi \int_0^{2\sqrt{\cos 2\varphi}} \sqrt{1+\frac{\rho^2}{4}}\,\rho\, d\rho = 2\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} d\varphi \int_1^{1+\cos 2\varphi} \sqrt{t}\, dt \\ &= \frac{4}{3}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left(t^{\frac{3}{2}}\Big|_1^{1+\cos 2\varphi}\right) d\varphi = \frac{4}{3}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left(t^{\frac{3}{2}}\Big|_1^{2\cos^2\varphi}\right) d\varphi = \frac{4}{3}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left(2\sqrt{2}\cos^3\varphi - 1\right) d\varphi \\ &= \frac{8\sqrt{2}}{3}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos^3\varphi\, d\varphi - \frac{4}{3}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} d\varphi = \frac{8\sqrt{2}}{3}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left(1-\sin^2\varphi\right) d(\sin\varphi) - \frac{2\pi}{3} \\ &= \frac{8\sqrt{2}}{3}\left(\sin\varphi - \frac{\sin^3\varphi}{3}\right)\Bigg|_{-\frac{\pi}{4}}^{\frac{\pi}{4}} - \frac{2\pi}{3} = \frac{40}{9} - \frac{2\pi}{3}. \end{aligned}$$