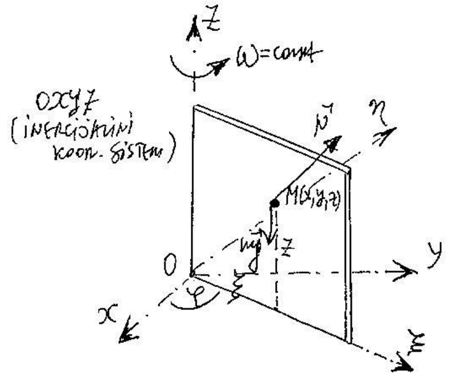
Primena Lagranževih jednačina druge vrste
na
(1) relatvno kretanje materijalne tačke:

1. Materijalna tačka M, mase m, može da se kreće bez trenja po ravni ξ0z. Ravan ξ0z obrće se oko nepomične ose oz konstantnom ugaonom brzinom ω. U početnom trenutku $t_o=0$, tačka M je bila u miru (u odnosu na ravan ξ0z) u položaju $M_o(\xi_o=5,\ z_o=0)$. Odrediti opšte jednačine kretanja tačke M u odnosu na pokretni koordinatni sistem ξ0z. Uzeti da su generalisane koordinate: ξ, z.

$$\left\{ \frac{d}{dt}\frac{\partial T}{\partial \dot q_\alpha} - \frac{\partial T}{\partial q_\alpha} = Q_\alpha \right\}$$



$$T = \frac{1}{2} m V^2_{M_{aps}} = \frac{1}{2} m(\dot x^2 + \dot y^2 + \dot z^2)$$

$$\begin{cases} q_1 = \xi \\ q_2 = z \end{cases} \quad \begin{cases} x = \xi\cos\varphi & \dot x = \dot\xi\cos\varphi - \xi\dot\varphi\sin\varphi \\ y = \xi\sin\varphi & \dot y = \dot\xi\sin\varphi + \xi\dot\varphi\cos\varphi \\ z = z & \dot z = \dot z \end{cases}$$

$$T = \frac{1}{2} m(\dot\xi^2 + \xi^2\dot\varphi^2 + \dot z^2)$$

$$\frac{\partial T}{\partial \dot\xi} = m\dot\xi \quad \frac{\partial T}{\partial \xi} = m\xi\omega^2 \quad \frac{\partial T}{\partial \dot z} = m\dot z \quad \frac{\partial T}{\partial z} = 0$$

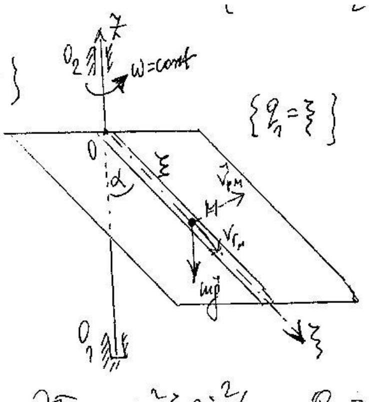
$$\delta A = -mg\,\delta z \quad \{ Q_\xi = 0,\ Q_z = -mg \}$$

$$\begin{cases} m\ddot\xi - m\omega^2\xi = 0 \\ m\ddot z - 0 = -mg \end{cases}$$

$$M: \begin{cases} \xi = C_1 \operatorname{ch}\omega t + C_2 \operatorname{sh}\omega t \\ z = -\frac{gt^2}{2} + C_3 t + C_4 \end{cases}$$

2. Ploča (u kojoj je izdubljen glatki kanal) zavarena je pod uglom α (α=const) za vertikalnu osovinu i obrće se oko vertikalne nepomične ose $0_1 0_2$ konstantnom ugaonom brzinom ω. Kroz kanal može da se kreće tačka M mase m. Uzeti da je generalisna koordinata ξ (za kanal je vezana pokretna osa 0ξ), pa odrediti opštu jednačinu relativnog kretanja tačke M, tj. ξ (t)=?

$\{\alpha = const\}$



$\{ q_1 = \xi \}$

$$T = \frac{1}{2} m V^2_{M_{aps}}$$

$$V_{r_M} = \dot\xi \qquad V_{pM} = \xi\omega\sin\alpha \qquad (\vec V_{r_M} \perp \vec V_{p_M})$$

$$T = \frac{1}{2} m(\dot\xi^2 + \omega^2\xi^2\sin^2\alpha)$$

$$-z = \cos\alpha\,\xi$$

$$-\delta z = \cos\alpha\,\delta\xi$$

$$\delta A = -mg\,\delta z = mg\cos\alpha\,\delta\xi$$

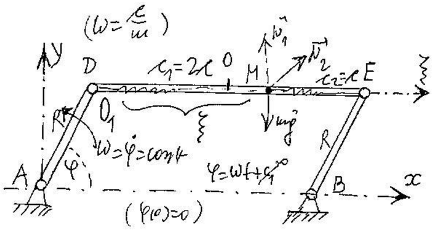
$$\frac{\partial T}{\partial \dot\xi} = m\dot\xi \quad \frac{\partial T}{\partial \xi} = m\omega^2\xi\sin^2\alpha \quad Q_\xi = mg\cos\alpha \quad m\ddot\xi - (m\omega^2\sin^2\alpha)\xi = mg\cos\alpha$$

$$\ddot\xi - (\omega^2\sin^2\alpha)\xi = g\cos\alpha$$

$$\left\{ \frac{d}{dt}\frac{\partial T}{\partial \dot\xi} - \frac{\partial T}{\partial \xi} = Q_\xi \right\}$$

$$M: \left\{ \xi = C_1 \operatorname{ch}(\omega\sin\alpha\, t) + C_2 \operatorname{sh}(\omega\sin\alpha\, t) - \frac{g\cos\alpha}{(\omega\sin\alpha)^2} \right.$$

3. Krivaja AD (AD=BE=R) obrće se konstantnom ugaonom brzinom ω i dovodi u kretanje cev DE (DE=2R, DO=OE=R). Veze u tačkama A, B, D i E su zglobne. Unutar glatke cevi DE može da se kreće tačka M mase m, vezana za dve opruge DM i ME, krutosti $c_1=2c$ $c_2=c$. U $t_o=0$ tačka M je u koordinatnom početku relativne koordinatne ose 0ξ s relativnom brzinom $R\omega_o$ (opruge su tada bile nenapregnute); odrediti opštu jednačinu relativnog kretanja tačke M, tj. ξ (t)=? Uzeti da je generalisna koordinata ξ. Sistem je u vertikalnoj ravni, $\varphi(0)=0$.



$\{q_1=\xi\}$ $\qquad \frac{d}{dt}\frac{\partial T}{\partial \dot\xi}-\frac{\partial T}{\partial \xi}=Q_\xi$

$T=\frac{1}{2}mV_M^2=\frac{1}{2}m\left(\dot\xi^2+R^2\dot\varphi^2-2R\dot\varphi\dot\xi\sin\varphi\right)$

$\frac{\partial T}{\partial \dot\xi}=\dot\xi m-mR\dot\varphi\sin\varphi$

{ Axy inercijani koor. sistem }

$M:\begin{cases} x=R\cos\varphi+\xi \\ y=R\sin\varphi \end{cases}$ $\qquad \begin{cases} \dot x=-R\dot\varphi\sin\varphi+\dot\xi \\ \dot y=R\dot\varphi\cos\varphi \end{cases}$

$\delta A=-F_{e1}\delta\xi-F_{e2}\delta\xi$ $\qquad F_{e1}=c_1\xi$ $\quad F_{e2}=c_2\xi$

$Q_\xi=-3c\xi$

$\frac{d}{dt}\frac{\partial T}{\partial \dot\xi}=m\ddot\xi-mR\underbrace{\ddot\varphi}_{0}\sin\varphi-mR\dot\varphi^2\cos\varphi$

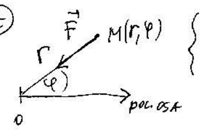
$m\ddot\xi-mR\dot\varphi^2\cos\varphi-0=-3c\xi$ $\qquad \ddot\xi+\left(\frac{3c}{m}\right)\xi=R\dot\varphi^2\cos(\omega t)$ $\qquad \left(k^2=\frac{3c}{m}\right)$

$\xi=C_1\cos\sqrt{\frac{3c}{m}}\,t+C_2\sin\sqrt{\frac{3c}{m}}+\frac{mR\omega^2}{2c}\cos(\omega t)$

Primena Lagranževih jednačina druge vrste
na
(2) kretanje materijalne tačke pod dejstvom centralne sile:

1. Materijalna tačka M jedinične mase kreće se pod dejstvom centralne privlačne sile čiji je intenzitet $F=\frac{mk^2}{r^3}$. Odrediti za tačku M, u odnosu na date generalisane koordinate r, φ: 1) diferencijalne i konačne jednačine kretanja, 2) putanju. U početnom trenutku $t_0=0$, $r_0=2$, $\varphi_0=0$, $V_0=1/2$, a ugao između brzine i potega bio 45°. Neka je k=1; zadate veličine su date u osnovnim jedinicama SI sistema.

(t)

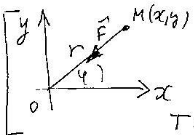


$$\begin{cases} q_1 = r \\ q_2 = \varphi \end{cases}$$

$$(1)\ \frac{d}{dt}\frac{\partial T}{\partial \dot r} - \frac{\partial T}{\partial r} = Q_r \qquad (2)\ \frac{d}{dt}\frac{\partial T}{\partial \dot\varphi} - \cancel{\frac{\partial T}{\partial \varphi}}^{\,0} = \cancel{Q_\varphi}^{\,0} \qquad (m=1;\ k=1)$$

$$dA = \vec F \cdot d\vec r = \left[\frac{f(r)}{}\left(\frac{\vec r}{r}\right)\right]\cdot d\vec r = f(r)\,dr \qquad \begin{cases} Q_r = f(r) = -\frac{mk^2}{r^3} = -\frac{1}{r^3} \\ Q_\varphi = 0 \end{cases}$$

$$T = \frac{1}{2}m\left(\dot r^2 + r^2\dot\varphi^2\right)$$



$$\begin{cases} x = r\cos\varphi & \dot x = \dot r\cos\varphi - r\dot\varphi\sin\varphi \\ y = r\sin\varphi & \dot y = \dot r\sin\varphi + r\dot\varphi\cos\varphi \end{cases}$$

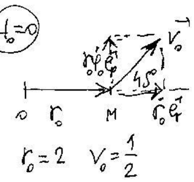
$$T = \frac{1}{2}m(\dot x^2 + \dot y^2) = \frac{1}{2}m\left(\dot r^2 + r^2\dot\varphi^2\right)$$

$$\frac{\partial T}{\partial \dot r} = m\dot r \qquad \frac{\partial T}{\partial r} = mr\dot\varphi^2$$

$$\frac{\partial T}{\partial \dot\varphi} = mr^2\dot\varphi \qquad \frac{\partial T}{\partial \varphi} = 0$$

$$\begin{cases} (1)\ \ m\ddot r - mr\dot\varphi^2 = -\frac{1}{r^3} \\ (2)\ \ \frac{d}{dt}\left(mr^2\dot\varphi\right) - 0 = 0 \end{cases} \Rightarrow r^2\dot\varphi = r_0^2\dot\varphi_0 = \frac{\sqrt2}{2} \Rightarrow \dot\varphi = \frac{\sqrt2}{2r^2} \Rightarrow (1)$$

($t_0=0$)



$r_0=2 \quad V_0=\frac{1}{2}$

$$\begin{cases} \dot r_0 = V_0\frac{\sqrt2}{2} \\ r_0\dot\varphi_0 = V_0\frac{\sqrt2}{2} \quad \dot\varphi_0 = \frac{\sqrt2}{8} \end{cases}$$

$$m=1 \qquad \ddot r - r\left(\frac{\sqrt2}{2r^2}\right)^2 = -\frac{1}{r^3}$$

$$\ddot r = -\frac{1}{2r^3} \qquad \ddot r = \frac{d}{dt}\left(\frac{dt}{dt}\right)\frac{dr}{dr} = \dot r\frac{d\dot r}{dr}$$

$$\int \dot r\, d\dot r = -\int\frac{dr}{2r^3} \qquad \frac{\dot r^2}{2} = \frac{1}{4r^2} + C_1 \qquad \frac{\dot r_0^2}{2} = \frac{1}{4r_0^2} + C_1 \Rightarrow C_1 = 0$$

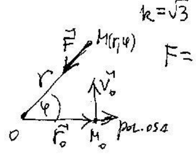
$$\dot r^2 = \frac{1}{2r^2} \qquad \dot r = \frac{1}{\sqrt2\, r} \qquad \int r\,dr = \int\frac{dt}{\sqrt2} \qquad \frac{r^2}{2} = \frac{t}{\sqrt2} + C_2 \qquad \frac{r_0^2}{2} = 0 + C_2 \qquad C_2 = 2$$

$$r^2 = \sqrt2 t + 4 \qquad \dot\varphi = \frac{\sqrt2}{2r^2} \qquad \int d\varphi = \int\frac{\sqrt2\,dt}{2(\sqrt2 t + 4)}$$

$$\begin{cases} \varphi = \frac{1}{2}\ln\left(2t + 4\sqrt2\right) \\ r = \sqrt{\sqrt2 t + 4} \end{cases} \Rightarrow \boxed{r^2 = e^{2\varphi}}$$

2. Materijalna tačka M mase m kreće se pod dejstvom centralne privlačne sile čiji je intenzitet $F=\frac{mk^2}{r^7}$. Odrediti za tačku M, u odnosu na date generalisane koordinate r, φ: 1) diferencijalne i konačne jednačine kretanja, 2) putanju. U početnom trenutku $t_0=0$, $r_0=1$, $\varphi_0=0$, $V_0=1$, a ugao između brzine i potega bio 90°. Neka je $k^2=3$; zadate veličine su date u osnovnim jedinicama SI sistema.

$k=\sqrt{3}$



$F=\frac{mk^2}{r^7}$

$(1)\ \frac{d}{dt}\frac{\partial T}{\partial \dot r}-\frac{\partial T}{\partial r}=Q_r \qquad (2)\ \frac{d}{dt}\left(\frac{\partial T}{\partial \dot\varphi}\right)-\cancel{\left(\frac{\partial T}{\partial \varphi}\right)}^{0}=\cancel{Q_\varphi}^{0} \Rightarrow \frac{\partial T}{\partial \dot\varphi}=const$

$dA=f(r)dr \quad \begin{cases} Q_r=f(r)=-\frac{mk^2}{r^7}=-\frac{\sqrt{3}^2 m}{r^7} \\ Q_\varphi=0 \end{cases}$

$\begin{cases} V_0=1 \quad r_0=1 \quad \varphi_0=0 \\ \dot r_0=0 \quad r_0\dot\varphi_0=V_0 \Rightarrow \dot\varphi_0=1 \end{cases}$

$T=\frac{1}{2}m(\dot r^2+r^2\dot\varphi^2) \qquad \frac{\partial T}{\partial \dot r}=m\dot r \qquad \frac{\partial T}{\partial r}=mr\dot\varphi^2 \qquad \frac{\partial T}{\partial \dot\varphi}=m\dot\varphi r^2 \qquad \frac{\partial T}{\partial \varphi}=0$

$\begin{cases} (1)\ m\ddot r-mr\dot\varphi^2=-\frac{m\sqrt{3}^2}{r^7} \\ (2)\ r^2\dot\varphi=r_0^2\dot\varphi_0=1 \quad \dot\varphi=\frac{1}{r^2} \end{cases} \qquad \ddot r-r\left(\frac{1}{r^2}\right)^2=-\frac{\sqrt{3}^2}{r^7} \Bigg| \Rightarrow \int \dot r d\dot r=\int\left(\frac{1}{r^3}-\frac{3}{r^7}\right)dr$

$\frac{\dot r^2}{2}=-\frac{1}{2r^2}+\frac{1}{2r^6}+c_1 \quad (c_1=0) \qquad \dot r^2=\frac{(1-r^4)}{r^6} \Rightarrow \int\frac{r^3}{\sqrt{1-r^4}}dr=\int dt$

$r^4=1-4t^2+c_2 \quad (r_0^4=1-0+c_2 \Rightarrow c_2=0)$

$(2)\Rightarrow \int d\varphi=\int\frac{dt}{\sqrt{1-4t^2}}$

$\begin{cases} r^4=1-4t^2 \\ \varphi=\frac{1}{2}\arcsin(2t)+\cancel{c_3}^{0} \end{cases} \Bigg| \Rightarrow \boxed{r^2=\cos 2\varphi}$

$\sin(2\varphi)=2t \quad r^4=1-(\sin 2\varphi)^2 \Rightarrow$